INTRODUCTION TO BICATEGORIES

Jean Bénabou

Part I

Introduction. This is the first part of a work concerned with the study of the following type of structure: A family of categories $\underline{S}(A, B)$ (A, B in a set $\underline{S}_o$) together with pairing functors $c(A, B, C): \underline{S}(A, B) \times \underline{S}(B, C) \to \underline{S}(A, C)$ which up to given coherent isomorphisms behave as if the $\underline{S}(A, B)$ were the $\mathrm{Hom}_?(B, A)$ for some "category" ?. The best known cases are perhaps $\underline{S}_o$ = one point, then we have a single category $\underline{S}$ with a multiplication in the sense of [B. 1], or a 2-category [B. 3] where the associativity isomorphisms are identities, or $\underline{S}_o$ = a set of rings, $\underline{S}(A, B)$ = category of (A, B)-Bimodules and $c(A, B, C) = \otimes_B$.

In §1 we formalise this situation in the definition of bicategory and show in §2 that many other cases considered by Epstein [E] or Yoneda [Y] fit in this pattern.

Even more important is the notion of morphisms defined in §4 where we do not require the functors $F(A, B): \underline{S}(A, B) \to \overline{\underline{S}}(\overline{A}, \overline{B})$ to commute with the $c(A, B, C)$, not even up to isomorphisms. The justification for such an apparently too complicated and unnecessarily general definition is in the number of examples (see §5) ranging from monads to pseudo-functors of [Gr] which can be handled and in the fact that most of the results ex-

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pected for the strict homomorphisms, hold for general morphisms, and have meaningful interpretation (§ 6).

In § 7 we define some of the invariants of a bicategory: the Poincaré and classifying categories and the Picard groupoid which will be used in Part II. Finally § 8 is devoted to the construction of the analogue of the path space, namely the bicategory of cylinders, which gives the possibility to define transformations between morphisms (similar to natural transformations, or homotopies). For this construction we have used heavily the geometrical analogy without which definitions and results seem artificial and are incomprehensible. In many cases we have even replaced the proofs -- essentially setting up very big commutative diagrams-- by more suggestive pictures.

In Part II, we will first complete the construction of the 3-dimensional part of Bicat, by defining "modifications" between transformations, then study the notions of representability, adjointness and equivalence, which are quite different in the two-dimensional case from their ordinary analogue. Then we will examine the case when the functors $c(A, B, C)$ have a right adjoint, and finally study many examples of bicategories, devoting the greatest time to bicategories of "Profunctors".