INTRODUCTION TO THE ALGEBRAIC THEORY OF POSITIVE CHARACTERISTIC
DIFFERENTIAL GEOMETRY

By M. SWEEDLER

What follows is based upon the introduction to a manuscript by M. Takeuchi and myself, [ST].

[H] and [Wa] present algebraic aspects of differential geometry very neatly. They naturally lead to speculation about the algebraic geometric analog of results therein. In many cases the algebraic geometric analogs are well known, especially in characteristic zero. In this paper we present algebraic results in positive characteristic which are motivated by methods of differential geometry involving the module of one forms, vector fields and the DeRham complex. One of our results is related to the two versions of Frobenius' theorem on integral submanifolds, [Wa, p.48,(1.64)] and [Wa,p.75, (3.32)], which alternatively depend on distributions or differential ideals in the DeRham complex. Our other result is related to the Poincaré lemma [Wa, p.156, Corollary(b)].

In attempting to transport ideas from differential geometry to positive characteristic algebraic geometry there are two problems to deal with. The first is that derivations vanish on $p^th$ powers in positive characteristic. In algebraic geometry the Kähler module is the analog of the module of one forms. In positive characteristic $p$ the Kähler module is rather myopic since derivations vanish on $p^th$ powers, and the derivation from an algebra to its Kähler module is of vital concern. Therefore at times it is necessary to work with algebras whose $p^th$ powers lie in the base ring. For our strongest results we work with a field
whose $p^{th}$ powers lie in a base field, i.e. a purely inseparable exponent one field extension.

The second problem is the DeRham complex itself. While there is a perfectly well defined algebraic analog of the DeRham complex, [0, p.155, Lemma 9.2], it too is myopic. Even for purely inseparable exponent one field extensions this is true as we shall show by examples. Instead of the algebraic DeRham complex we develop a new complex \{Q_*\} which is only defined in positive characteristic. \{Q_*\} is similar to the algebraic DeRham complex in that $Q_0$ is the algebra and $Q_1$ is the Kähler module. However for $i \geq 2$, $Q_i$ is a graded part of a universal commutative —not skew commutative— differential graded algebra.

**FROBENIUS, JACOBSON, GERSTENHABER**

The Frobenius theorem is about integral submanifolds of a manifold. It can be formulated in terms of distributions or differential ideals. If $A$ is the algebra of $C^\infty$ functions on a differentiable manifold then $\text{Der} A$, the set of derivations from the algebra $A$ to itself, is the set of vector fields on the manifold, [H, p.9, Definition]. As usual $\text{Der} A$ is a Lie algebra under Lie bracket and also a left $A$ module. A smooth distribution is a geometrically locally free $A$ submodule of $\text{Der} A$, [Wa, p.41, Definition 1.56]. The distribution is called involutive if it is a Lie subalgebra of $\text{Der} A$, [Wa, p.42, Definition 1.56]. $Q$ defined as $\text{Hom}_A(\text{Der} A, A)$ —the $A$ dual to $\text{Der} A$— is the module of one forms on the manifold. It is the degree one term of the DeRham complex. Let

$$A \xrightarrow{q} Q \xrightarrow{q'} Q \xrightarrow{q}$$

be the differential of the DeRham complex from its degree zero to degree one to degree two terms. Implicit in the two formulations of the Frobenius theorem is the result that a smooth distribution $L \subseteq \text{Der} A$ is a Lie subalgebra if and only if the ideal generated by $L^1$ in the DeRham complex is a differential ideal.