We show how the K-groups for the crossed product of a C*-algebra by an action of Z₂ are related to those of the fixed point algebra and of the ideal in the fixed point algebra generated by products of elements in the (-1) - eigenspace.

For a fixed locally compact group G, the construction that associates to a C*-dynamical system (A, G, a) the crossed product Ax G [6] is a functor from (G-dynamical systems ; equivariant *-homomorphisms) to (C*-algebras ; *-homomorphisms). This functor is sufficiently well-behaved that its composition with Kg for C*-algebras yields a pair of functors obeying equivariant versions of the rules for ordinary K-theory. (We remark that when G is compact and A is abelian, Kg(Ax G) coincides with the Atiyah-Segal topological G-equivariant K-theory of the spectrum of A [8] [5] [4].) It is thus feasible to compute Kg(Ax G) on an ad hoc basis in many instances. For special choices of G (e.g. IF, JR), there are also elegant results relating Kg(Ax G) to Kg(A) [7] [3], but in general the sort of information in terms of which the K-groups of the crossed product can most conveniently be computed will depend on what sort of group G is. The treatment below of the case G = Z₂ illustrates what happens for finite cyclic groups except that here the bookkeeping complications are minimal.
1. Exact sequence for $K_i(A \times \alpha \mathbb{Z}_2)$.

Henceforth, $A$ will be a $C^*$-algebra and $\alpha$ will be an automorphism of $A$ such that $\alpha^2 = \text{id}_A$. We let $A_0 = \{a \in A : \alpha(a) = a\}$, $A_1 = \{a \in A : \alpha(a) = -a\}$, and $J = A_1^2$ (the closed linear span of $\{xy : x, y \in A_1\}$). Notice that $A = A_0 + A_1$ via the projections $E_i : A \to A_i$ defined by $E_i(x) = \frac{1}{2}(x + (-1)^i\alpha(x))$ ($i = 0, 1$). Also, $A_1$ is a 2-sided $A_0$-module, $J$ is a closed 2-sided ideal of $A_0$, and $A_1 + J$ is a closed 2-sided ideal of $A$. The crossed product $A \times \alpha \mathbb{Z}_2$ consists of functions $f : \mathbb{Z}_2 \to A$ with multiplication $(fg)(0) = f(0)g(0) + f(1)\alpha(g(i+1))$ and involution $f^*(1) = \alpha^1(f(1))^*$ ($i = 0, 1$).

It is straightforward to check that the map

$$
\begin{pmatrix}
E_0(f(0) + f(1)) & E_1(f(0) - f(1)) \\
E_1(f(0) + f(1)) & E_0(f(0) - f(1))
\end{pmatrix}
$$

is an isomorphism of $A \times \alpha \mathbb{Z}_2$ with the $C^*$-subalgebra $\begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}$ of $A \otimes M_2$.

In the theorem below $K$ is the algebra of compact operators, and $M(\cdot)$ denotes the multiplier algebra. The maps $\iota_* : K_j(J) \to K_j(A_0)$ and $\partial : K_j(A_0/J) \to K_{j-1}(J)$ come from the cyclic exact sequence of $K$-groups produced by

$$
0 \to J \to A_0 \to A_0/J \to 0.
$$

(For this and related matters, see [9].)

As will be shown in the proof of the theorem, the maps $K_j(A_0) \to K_j(A_0 \times \alpha \mathbb{Z}_2)$ come from

$$
\begin{pmatrix}
0 & 0 \\
0 & A_0
\end{pmatrix} \to 
\begin{pmatrix}
A_0 & A_1 \\
A_1 & A_0
\end{pmatrix}.
$$