CHAPTER C-II

CHARACTERIZATION

OF POSITIVE SEMIGROUPS

ON BANACH LATTICES

by

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In this chapter our first goal is to find conditions on a generator $A$ of a semigroup $(T(t))_{t \geq 0}$ which are equivalent to the positivity of the semigroup. After the preparations in A-II, Sec. 2 this is easy if in addition we ask that the semigroup be contractive: $T(t)$ is a positive contraction for all $t \geq 0$ if and only if $A$ is dispersive (Section 1). For arbitrary (not necessarily contractive) semigroups a condition on the generator had been found in the case when $E = C(K)$ ($K$ compact), namely the positive minimum principle (P) (see B-II). One may easily reformulate this condition in arbitrary Banach lattices and show its necessity. However, only in special cases (for example if $A$ is bounded (see Section 1)) the positive minimum principle is sufficient for the positivity of the semigroup. In fact, on $L^2(\mathbb{R})$ there exists a non-positive semigroup whose generator satisfies (P) (Section 3).

Looking for another condition we consider the Laplacian $\Delta$ as a prototype. Defined on a suitable domain, $\Delta$ generates a positive semigroup on $L^p(\mathbb{R}^n)$. Kato proved the following distributional inequality for the Laplacian:

$$(\text{sign } f) \Delta f \leq \Delta |f|$$

for all $f \in L^1_{\text{loc}}$ such that $\Delta f \in L^1_{\text{loc}}$. In Section 3 we will show that an abstract version of Kato's inequality for a generator $A$ together with an additional condition is equivalent to the positivity of the semigroup generated by $A$.

Domination of one semigroup by another can be characterized by an analogous condition for the generators (Section 4). The results will be applied to Schrödinger operators on $L^p(\mathbb{R}^n)$. 
Finally, in Section 5 we show that \((T(t))_{t \geq 0}\) is a lattice semigroup (i.e., \(|T(t)f| = T(t)|f|\) for all \(t \geq 0\), \(f \in E\)) if and only if \(A\) satisfies Kato's equality. This parallels the case when \(E = C_0(X)\), but if \(E\) has order continuous norm the strong form of Kato's equality can be considered (in particular, \(f \in D(A)\) implies \(|f| \in D(A)\) if \(A\) is the generator of such a semigroup).

1. POSITIVE CONTRACTION SEMIGROUPS AND BOUNDED GENERATORS

In this section we first characterize generators of positive contraction semigroups on a real Banach lattice \(E\).

For that we use the results developed in A-II, Section 2 for the canonical half-norm \(N^+ : E \to \mathbb{R}\) given by

\[
N^+(f) = \|f^+\| \quad (f \in E).
\]

**Remark.** It is easy to see that \(N^+(f) = \inf \{\|f+g\| : g \in E_+\} = \text{dist} (-f,E_+)\) (cf. A-II, Rem. 2.8).

It is obvious that \(N^+\) is a strict half-norm (see A-II, (2.12)).

The subdifferential of \(N^+\) is given by

\[
dN^+(f) = \{\phi \in E_+^1 : \|\phi\| \leq 1, \langle f, \phi \rangle = \|f^+\|\}
\]

(this follows from the definition, see A-II, (2.5)).

**Examples 1.1.**

a) Let \(E = C_0(X)\) (\(X\) locally compact). Let \(f \in E\). There exists \(x \in X\) such that \(f(x) = \|f^+\|_\infty\). Then \(\delta_x \in dN^+(f)\).

b) Let \(E = L^p(X,\Sigma,\mu)\), where \((X,\Sigma,\mu)\) is a \(\sigma\)-finite measure space and \(1 < p < \infty\). Let \(f \in E\) satisfy \(f^+ \neq 0\). Let

\[
\phi(x) = \begin{cases} 
c \cdot f(x)^{p-1} & \text{if } f(x) > 0 \\
0 & \text{if } f(x) \leq 0
\end{cases}
\]

where \(c > 0\) is such that \(\int |f(x)|\phi(x)\,dx = \|f^+\|\).

Then \(dN^+(f) = \{\phi\}\).

c) Let \(E = L^1(X,\Sigma,\mu)\), where \((X,\Sigma,\mu)\) is a \(\sigma\)-finite measure space, and \(f \in E\). Let \(\phi \in L^{\infty}(X,\Sigma,\mu)_+\). Then \(\phi \in dN^+(f)\) if and only if

\[
\begin{align*}
\phi(x) &= 1 & & \text{if } f(x) > 0, \\
0 \leq \phi(x) \leq 1 & & \text{if } f(x) = 0 \text{ and } \\
\phi(x) &= 0 & & \text{if } f(x) < 0.
\end{align*}
\]