§0. Introduction

Some limit theorems for Minkowski sums of independent identically distributed random compact convex (c.c.) subsets of a separable Banach space $B$ were obtained in [13]. (See also [6], [16], [21], [22] and [24] for the classical Gaussian case.) The central limit theorems proved there did not presuppose the existence of a limiting probability law on random sets: in fact we only considered convergence in distribution of the real random variables

$$\{a_n^m(S_n/n, EX)\}_{n=1}^\infty.$$  

(Here $\delta$ is Hausdorff distance, $S_n = \sum_{j=1}^n X_i$ is the Minkowski sum of the i.i.d. random c.c. sets $X_i$ with law $L(X_i) = L(X)$, $EX$ is the expectation of the random set $X$ and $a_n \in \mathbb{R}$, $a_n \to \infty$. See these and other definitions below.) In other words, those were speed of convergence results for the law of large numbers. The next natural question is whether the laws of the sums $L(S_n/c_n)$ can be approximated by Gaussian, stable and, in general, by $M$-infinitely divisible c.c. sets (with $M$ for Minkowski addition). This question is obviously related to (in fact dependent upon) the existence of a Lévy-Khinchin representation for $M$-infinitely divisible (M-i.d.) random c.c. sets.

In this paper we survey recent work on M-i.d. c.c. sets by Lyashenko [17], Mase [18], Vitale [23] and ourselves [11], [12], and present as well several partial results and examples about general M-i.d. c.c. sets in infinite-dimensional Banach spaces.

The situation is as follows: (1) the M-i.d. c.c. sets of $\mathbb{R}^d$ are completely determined; (2) the $p$-stable c.c. sets of any separable infinite-dimensional Banach space $B$ are also completely characterized (and the results show that, curiously enough, often a CLT for $\{a_n^m(S_n/n, EX)\}$ holds but no non-degenerate limiting stable c.c. set exists for the sums); and (3), very little is known about general M-i.d. c.c. sets in infinite dimensions.

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By using support processes these problems reduce to special questions about infinitely divisible laws in the Banach space of continuous functions on a compact metric space. The available general theory in this setting turns out to be very useful.

Our interest in this subject stems from the facts that it is a fertile field for application of Banach space probability theory, and that some of the results are quite elegant. Moreover, we know of at least one interesting application of limit theorems for random sets: see Artstein and Hart [3] and Artstein [2], where a central limit theorem and a law of large numbers for Minkowski addition are applied in an optimization problem of interest in economics. (However, it may be argued that this theory has not yet had significant interactions with other areas.)

This paper is organized as follows. Section 1 provides a brief survey of the known results including support processes, M-i.d. c.c. subsets of $\mathbb{R}^d$ and p-stable c.c. subsets of separable Banach spaces. Section 2 contains new results on M-i.d. c.c. sets of infinite-dimensional Banach spaces essentially showing the inadequacy of the methods used in $\mathbb{R}^d$. Some necessary conditions for M-i.d. are provided as well as some examples and counterexamples. It is shown in particular that it is impossible, in infinite dimensions, to choose a point from every compact convex subset in a linear and uniformly continuous way i.e., there are no "Steiner points" defined for all c.c. sets in infinite dimensions. (The Steiner point is an important tool in finite dimensions.)

Now we describe the notation and basic definitions used throughout. Let $B$ denote a separable Banach space with norm $\| \cdot \|$, and let $K(B)$ be the collection of nonempty compact subsets of $B$. Define on $K(B)$ two basic operations:

$$
A+C := \{a+c: a \in A, c \in C\} \text{ (Minkowski addition)}
$$

$$
aA := \{a\alpha: a \in A\}, \quad \alpha > 0 \text{ (positive homothetics)}
$$

where $A, C \in K(B)$. $K(B)$ is not a vector space since generally $A+(-A) \neq \{0\}$. However, $K(B)$ becomes a complete separable metric space when endowed with the Hausdorff distance $\delta$,

$$
\delta(A,C) := \max\{\sup_{a \in A} \inf_{c \in C} \|a-c\|, \sup_{c \in C} \inf_{a \in A} \|a-c\|\}
$$

$$
= \inf\{\varepsilon > 0: A \subset C^\varepsilon, C \subset A^\varepsilon\}
$$

where $D^\varepsilon := \{x \in B: \delta(x,D) < \varepsilon\}$. Let