Abstract. In an earlier paper, we proved that a plot of $\log n$ independent Brownian motions in dimension $d = 1$, for times in $[0, n]$, is nearly certain to give the appearance of a shaded region with square root boundaries, when subjected to the rescaling of the functional iterated logarithm law. Here we prove that for every finite dimension $d$ the same conclusion holds if just one point is plotted from each of $\log n$ Brownian paths having variance parameter equal to 1, provided these points are selected uniformly in the time interval $[0, n]$ and independently of the paths.

Introduction. Our earlier result may be visualized by thinking of a smoke stack which, at time zero, emits a large number $\log n$ of burning particles whose gaseous trajectories are observed over the time interval $[0, n]$ as they drift in horizontally moving air. Our result suggested a plume which, when viewed from the side and appropriately rescaled, densely occupies a region with square root boundaries. In
effect, the \( \log n \) independent \( \mathbb{R}^1 \)-valued Brownian motions over times 
\([0, n]\), when rescaled, fill the space between the square root boundaries 
before they spill much outside them.

Our new result suggests the same type of plume if the particles 
leave no trails but are emitted at random times which are uniform over 
\([0, n]\). Moreover, the new result extends to an arbitrary finite dimen-
sion \( d \), when plotting one randomly selected point from each of \( \log n \)
independent \( \mathbb{R}^d \)-valued Brownian motions (having variance parameter 1) 
over time \([0, n]\). For convenience we prove the equivalent result for 
selecting one random point from each of \((\log n)^d\) independent \( \mathbb{R}^d \)-valued 
Brownian motions, having variance parameter \( d \), over time \([0, n]\).

Formulation. We use \( \ln x \) to denote \( \log_e x \). The evaluations of inde-
pendent \( \mathbb{R}^d \)-valued Brownian motions at any fixed times are independent 
normal random vectors. Fix an arbitrary \( c > 0 \) and finite dimension \( d \).

For each integer \( n \geq 3 \) consider the 'plume' given by

\[
\{(K_i, \sqrt{d} Y_i) : 1 \leq i \leq c_n \} = [c(n)^d],
\]

where \( Y_i \) are iid \( \mathcal{N}(0, 1) \) and the times \( K_1, K_2, \ldots \) are iid uniform 
on the integers 1, \ldots, \( n \), and independent of the \( Y_i \). We compare this 
plume with the 'iterated logarithm region',

\[
\{(t, y) : e \leq t \leq n, \|y\|_d \leq \sqrt{2d \ln t} \} \subset [e, n] \times \mathbb{R}^d.
\]

Our result makes the comparison after rescaling time by \( n \), and 
displacement \( \|y\|_d \) by \( a_n = \sqrt{2d n \ln n} \). Let

\[
T_{\ln} = K_i / n; Y_{\ln} = \sqrt{d} Y_i / a_n.
\]