SMALL EIGENVALUES OF THE LAPLACIAN
AND EXAMPLES

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O Introduction. Let \( (M, g) \) be a 2-dimensional, closed, connected, oriented Riemannian manifold with strictly positive Gaussian curvature \( K \).

\( K_0 := \min_K, \ K_1 := \max_K \). The differential equation \( \Delta f + \lambda f = 0 \) determines the eigenfunctions \( f \in C^\infty (M) \) and the eigenvalues \( \lambda = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \) of the Laplacian \( \Delta \). The eigenvalues of the sphere \( S^2 (K) \) are given by \( \lambda_\ell = \ell (\ell + 1) , \ell \in \mathbb{N} \), but for manifolds which are "not too far from the sphere" like an ellipsoid of revolution one does not even know the first eigenvalue \( \lambda_1 \).

The following estimates are sharp in the case that \( M \) is a sphere:

\[
\lambda_1 \in [2K_0, 2K_1] \quad [3] \quad [2] \quad , \quad \lambda \notin (2K_1, 6K_0) \quad [1].
\]

These results suggests investigation of eigenvalues \( \lambda \) with \( \lambda > 6K_1 \). The following theorem improves a result of Kozlowski [5].

1 Theorem ([7]). Let \( (M, g) \) be a 2-dimensional, closed, oriented, connected Riemannian manifold with strictly positive Gaussian curvature \( K \). Assume furthermore \( 6K_1 < 12K_0 \) and

\[
\max_M |\text{grad } K|^2 < \frac{4}{27} \left[ 10K_1^3 + 18K_0K_1^2 - 216K_0^2 (K_1 - K_0) \\
+ (7K_1^2 - 24K_1K_0 + 36K_0^2)^{3/2} \right].
\]

Then the polynomial

\[
P(\lambda) = \frac{1}{4}(\lambda - 12K_0)(\lambda - 6K_1)(\lambda - 2K_0) + \max_M |\text{grad } K|^2
\]

has two zeros \( a, b \), which fulfil \( 6K_1 \leq a < b \leq 12K_0 \) and no eigenvalue of the Laplacian lies in the interval \( (a, b) \).

The steps of the proof are:

1. Choise of a tensor \( A_{ijkl} \).

For related investigations see M. Kozlowski's article in this volume.
2. Calculation of $\int |A_{ijkl}|^2 \, d\omega$.

3. Application of integral formulas and estimates.

4. Discussion of the inequality $0 \leq \int |A_{ijkl}|^2 \, d\omega \leq P(\lambda) \int |f_1|^2 \, d\omega$ where $P(\lambda)$ is a polynomial in $\lambda$.

The last step promises sharp results only if $\|A_{ijkl}\|^2$ is as small as possible.

In chapter 2 of [6] it is outlined that this condition leads to the choice of a traceless and totally symmetric tensor $A_{ijkl}$.

In the case of constant Gaussian curvature $K$ the zeros of $P(\lambda)$ are just the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the sphere $S^2(K)$.

The resulting integral formula gives also a very short proof ([7], part 5) of a result of Kozlowski and Simon [6] concerning minimal immersions:

Let $(M^2,g)$ be a closed, connected manifold with curvature $K$,

$$\frac{1}{6} \leq K \leq \frac{1}{3}.$$ If $\tilde{x}: M^2 \to S^N(1)$ is an isometric minimal immersion, then there are only two possibilities: $K = \frac{1}{3}$ or $K = \frac{1}{6}$.

2 Preliminaries. Let $g^{ij}$ resp. $g_{ij}$ denote the components of the metric tensor $g$ resp. $g^{-1}$ in local coordinates. $\nabla$ and $d\omega$ denote the corresponding covariant differentiation and the volume element on $M$. The components $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$ have the sign of [4]. As usual raising and lowering of indices are defined. For $f \in C^\infty(M)$ $f_{ij} := \nabla_i \nabla_j f$ denotes the components of the Hessian and $\nabla f := g^{ij}f_{ij}$ denotes the Laplacian.

Finally we need the first four Ricci identities:

$$f_{ij} = f_{ij},$$

$$f_{ijk} = f_{ikj} + f_{h}R_{ijk}^{h},$$

$$f_{ijkl} = f_{ijlk} + f_{h}R_{ikl}^{h} + f_{ih}R_{jkl}^{h},$$

$$f_{ijklr} = f_{ijkrl} + f_{hjk}R_{ilr}^{h} + f_{ihk}R_{jlr}^{h} + f_{ijh}R_{klr}^{h}.$$