KRULL DIMENSION AND ARTIN ALGEBRAS

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Introduction

As is shown in [9] for a tame hereditary Artin Algebra $A$, the category $A$-mod, considered as a ring with several objects has Krull dimension two. This means that the category $F$ of all finitely presented contravariant functors from $A$-mod into the category of abelian groups admits a filtration of Serre subcategories $0 = F^{-1} \subset F_0 \subset F_1 \subset F_2 = F$ where $F_i / F_{i-1}$ is the category of all objects of finite length in the quotient category $F/F_{i-1}$.

There are several reasons to be interested in the Krull dimension of $A$-mod. The first is the study of exact sequences and morphisms in $A$-mod. The filtration of $F$ leads to a hierarchy of exact sequences, where the Auslander-Reiten sequences form the lowest level. This hierarchy is studied in [9] for the hereditary case. Another point is the characterization of the representation type by Krull dimension. A result of M. Auslander [1] shows that $K$-dim($A$-mod) = 0 if and only if $A$ is of finite representation type. Further, the Krull dimension of $A$-mod does not exist for many classes of algebras being of wild representation type [10]. Moreover, if $A$-mod has finite Krull dimension, say $n \in \mathbb{N}$, then $n$ is an upper bound for the pure global dimension of $A$ on either side [10].

Recently, H. Lenzing gave a geometrical explanation of the occurrence of Krull dimension two in the tame hereditary case [15].

The aim of this paper is to determine the Krull dimension of $A$-mod for further classes of tame Artin algebras.

In section 3 we obtain $K$-dim($A$-dim) = 2 for tame tilted algebras (Theorem 3.4), tame domestic one-relation algebras (Theorem 3.6.) and algebras which are stably equivalent to hereditary Artin algebras (Corollary 3.9).

An example in section 4 shows that there are non-domestic tame algebras $A$ for which $K$-dim($A$-mod) does not exist. Hence the existence of Krull dimension seems to be characteristic for the tame domestic representation type. However, this does not necessarily mean the useless of Krull dimension for the investigation of the non-domestic tame type.
Most of these results are consequences of Krull dimension two for the hereditary case using the general formalism of Krull dimension. This formalism will be developed in the first two sections. Here the most important point is to study when the existence of Krull dimension is transmitted to subcategories and factor categories. Our notion of Krull dimension is a variant of the Krull dimension introduced by P. Gabriel [7] for Grothendieck categories. We also investigate the connections of these two dimensions.

1. General properties of Krull-Gabriel dimension

In this section we study the Krull dimension of a small abelian category. The proofs for all the results in this section are done by transfinite induction. So we omit the proofs and refer to [8].

For convenience of the reader, we first recall some basic definitions and properties.

Let \( C \) be an abelian category and \( C' \) a Serre subcategory of \( C \). Then \( C/C' \) denotes the quotient category of \( C \) relative to \( C' \) [12]. Again, \( C/C' \) is abelian and the quotient functor \( T : \; C \rightarrow C/C' \) is exact [7].

A morphism \( \phi : TX \rightarrow TY \) in \( C/C' \) is of the form \( \phi = T(s_2)^{-1} \circ T(f) \circ T(s_1)^{-1} \) where \( s_1 : X' \rightarrow X \) is a monomorphism with cokernel in \( C' \), \( s_2 : Y \rightarrow Y' \) is an epimorphism with kernel in \( C' \) and \( f : X' \rightarrow Y' \) is a morphism in \( C' \). \( T(s_2)^{-1} \) is an isomorphism iff \( \ker f \) and \( \coker f \) are in \( C' \). Further, up to isomorphism, any exact sequence \( 0 \rightarrow TX \rightarrow TY \rightarrow TZ \rightarrow 0 \) in \( C/C' \) is the image under the quotient functor of an exact sequence in \( C \) with middle term \( Y \) [11]. Hence, an object \( X \) of \( C \) becomes simple in \( C/C' \) iff \( X \) is not an object of \( C \) and for each subobject \( U \) of \( X \) either \( U \) or \( X/U \) belong to \( C' \).

Now, let \( H : C \rightarrow \mathcal{D} \) be an exact functor and \( \ker H \) the Serre subcategory of all objects \( X \in C \) with \( H(X) = 0 \). For a Serre subcategory \( C' \subseteq \ker H \), there exists an unique exact functor \( \hat{H} : C/C' \rightarrow \mathcal{D} \), making the following diagram commutative [7]:

\[
\begin{array}{ccc}
C & \xrightarrow{T} & C/C' \\
\downarrow{H} & & \downarrow{\hat{H}} \\
\mathcal{D} & & \\
\end{array}
\]

\( \hat{H} \) is faithfull if and only if \( C' = \ker H \).

We are interested in the case \( \hat{H} \) being an equivalence. More generally, we get: