MEASURE-VALUED PROCESSES
CONSTRUCTION, QUALITATIVE BEHAVIOR AND STOCHASTIC GEOMETRY

Donald Dawson
Department of Mathematics and Statistics,
Carleton University,
Ottawa, Canada.

1. INTRODUCTION.

The objective of this paper is to review recent developments and applications of measure-valued stochastic processes. We begin by reviewing the construction and characterization of measure-valued processes and also illustrate the method of duality which is a useful tool in the study of some special classes of measure-valued processes. We then consider the qualitative behavior of probability-measure-valued processes, that is, processes in which the total measure is conserved. A classification of the qualitative behavior of these processes is introduced, namely, recurrent, ergodic, transient-diffusive and transient-wandering. In the next part we consider possibly infinite measure-valued processes including spatially homogeneous measure-valued processes. In this part we emphasize stochastic geometric aspects of the behavior of the processes. For example, we discuss the set-valued support process and questions involving clustering at both microscopic and macroscopic levels. Finally these questions are discussed in detail for measure-valued branching processes.

2. ORIGIN AND EXAMPLES OF MEASURE-VALUED DIFFUSION PROCESSES.

Beginning with the work of Feller (1951) the diffusion approximation has played an important role in stochastic population modelling. In particular in the diffusion limit certain questions concerning extinction and fixation probabilities can be answered in terms of Feller's boundary classification. In this section we consider the extension of the diffusion approximation to the space-time context. In this case the limiting diffusion is a measure-valued Markov process which is characterized as the solution of a measure-valued martingale problem. In order to introduce the notion of diffusion approximation for spatially distributed systems we present three simple but natural models which lead to these considerations.
MODEL I: POPULATION OF BRANCHING RANDOM WALKERS.

Consider a finite population of individuals who inhabit the one dimensional lattice $\mathbb{Z}^\varepsilon = \{0, \pm \varepsilon, \pm 2\varepsilon, \pm 3\varepsilon, \ldots\}$ where $\varepsilon > 0$ is a small parameter. We assume that there are initially $x(0)/\varepsilon$ individuals at the origin and that all individuals have mass $\varepsilon$. The dynamics are assumed to be continuous time binary branching and nearest neighbour symmetric random walk. The expected time between random walk jumps and also between binary fissions for each individual are assumed to be equal to $\varepsilon$. The branching offspring distribution is assumed to be given by:

Probability of death (0 offspring) = $(\frac{1}{2} - 2\alpha\varepsilon)$
Probability of two offspring = $(\frac{1}{2} + 2\alpha\varepsilon)$.

For $t \geq 0$ and $A \in \beta(R^1)$, the Borel subsets of $R^1$, $X^\varepsilon(t,A) :=$ Total mass in the set $A$ at time $t$.

Then $X^\varepsilon$ is a measure-valued Markov process. The total mass at time $t$ is $X^\varepsilon(t) := X(t,R^1)$.

Feller (1951) proved that as $\varepsilon \to 0$,

$x^\varepsilon(.) \to x(.)$ (weak convergence of processes)

where $x(.)$ is a one dimensional diffusion process with generator of the form:

$$2 \frac{d}{dx}\left( ax \frac{d}{dx} + \frac{1}{2} \gamma x^2 \frac{d^2}{dx^2}\right).$$

This process can also be characterized as the unique strong solution of the stochastic differential equation:

$$dx(t) = a x(t) dt + \gamma x^\frac{1}{2}(t) dw(t), \quad \gamma \geq 0,$$

where $w$ is a standard Brownian motion. Subsequently in population biology this equation has been generalized by the addition of "environmental stochasticity" and possibly non-linear growth leading up to the stochastic differential equation

$$dx(t) = f(x(t)) dt + \gamma_1 x^\frac{1}{2}(t) dw_1(t) + \gamma_2 x(t) dw_2(t)$$

where $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, and $w_1$ and $w_2$ are independent Brownian motions.

In order to describe at the heuristic level the diffusion limit as $\varepsilon \to 0$ of the spatially distributed model, let

$X(t,x) :=$ limiting density of the population at $x$ at time $t$. 
