ITERATION AND ANALYTIC CLASSIFICATION OF
LOCAL DIFFEOMORPHISMS OF $\mathbb{C}^v$

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§1. INTRODUCTION. RESURGENT FUNCTIONS

Two fundamental problems arise in connection with the local diffeomorphisms ("diffeos" for short) of $\mathbb{C}^v$: how to iterate them (formally or analytically) and how to classify them (again, formally or analytically). Actually, although those twin problems are largely equivalent, the second one is slightly more comprehensive. We shall therefore start with classification (§2 and §3) before tackling iteration (§4).

Everything rests on the use of resurgent functions. Roughly speaking these are divergent power series:

$$\phi(z) = \sum a_n z^{-n}$$

whose Borel transforms:

$$\mathcal{B}(\zeta) = \sum a_n \zeta^{n-1}/(n-1)!$$

converge in a neighbourhood of the origin and give rise, through analytic continuation in $\zeta$, to analytic functions which, though usually multivalued, never exhibit any cuts (that is to say, functions with isolated but otherwise arbitrary singularities). These functions are an algebra under the formal multiplication of the originals $\phi$ or, what amounts to the same, under the convolution $\ast$ of the Borel transforms:

$$\hat{\phi} \ast \hat{\psi}(\zeta) = \int_0^\zeta \hat{\phi}(\zeta_1)\hat{\psi}(\zeta-\zeta_1)d\zeta_1 \quad (\zeta \sim 0)$$

(1.1)

(1.2)

(1.3)
On this algebra one introduces a whole array of linear operators $\Delta_\omega$ which in some sense measure the singular behaviour of each $\phi$ at its singular points $\omega$ and which are so defined as to be derivations:

$$\Delta_\omega(\phi \ast \psi) = (\Delta_\omega \phi) \ast \psi + \phi \ast (\Delta_\omega \psi) \quad (1.4)$$

Or equivalently, keeping the same symbols $\Delta_\omega$ but reverting to the originals $\phi$:

$$\Delta_\omega(\phi \cdot \psi) = (\Delta_\omega \phi) \cdot \psi + \phi \cdot (\Delta_\omega \psi) \quad (1.5)$$

The operators $\Delta_\omega$ are known as alien derivations. They are subject to no a priori commutation constraints.

All these constructions extend to power series far more general than (1.1). For precise statements see [Ell].

§2. DIFFEOMORPHISMS TANGENT TO THE IDENTITY. THEIR FORMAL CLASSIFICATION

To illustrate the method, we shall deal with the case of local diffeos of $\mathbb{C}^\nu$ that are tangent of order $p(p \geq 1)$ to the identity, hence of the form:

$$f : x_i \to f_i(x) = x_i + o(x^p) \quad (1 \leq i \leq \nu) \quad (2.1)$$

More precisely, those diffeos may be written:

$$f : x_i \to f_i(x) = x_i + Q_i(x) + S_i(x) \quad (Q_i \in \mathbb{C}[x], S_i \in \mathbb{C}[x]) \quad (2.2)$$

the $Q_i(x)$ being homogeneous polynomials of degree $p+1$ in $x_1, \ldots, x_\nu$ and the $S_i(x)$ being functions holomorphic near $0$ and vanishing of order $p+1$ at $0$. The integer $p$ is said to be the level of the diffeo $f$.

The rational self-mapping

$$Q : \gamma = (\gamma_1, \ldots, \gamma_\nu) \to Q(\gamma) = (Q_1(\gamma), \ldots, Q_\nu(\gamma)) \quad (2.3)$$

of the projective space $\mathbb{P}_{\nu-1}(\mathbb{C})$ is known as directive mapping of $f$. Its fixed points are called eigenradii of $f$. For each eigenradius $\gamma$ one denotes:

$$1 - \lambda_1(\gamma), 1 - \lambda_2(\gamma), \ldots, 1 - \lambda_{\nu-1}(\gamma) \quad (2.4)$$