§ 10. Computation of characteristic numbers

In this section we establish a method for determining the (local and global) characteristic numbers of a condition. This is done by intersecting the condition with a suitable surface and by relating the characteristic numbers to numerical characters of the intersection curve.

10.1. Lemma

Given a condition \( K \), there exists a non-empty open set \( B_0 \subseteq B \) such that if \( S \) is an irreducible surface cutting \( B \) transversally at \( C \in B_0 \), then the restrictions \( \bar{X} \) and \( \bar{Y} \) of \( X \) and \( Y \) to \( S \), respectively, form a system of parameters for \( S \) in a neighbourhood of \( C \). Furthermore, if \( S \not\subseteq K \) then the Newton-Cramer polygon of the curve \( K \cap S \) with respect to the parameters \( \bar{X}, \bar{Y} \) coincides with the Newton-Cramer polygon of \( K \).

Proof

With the same notations as in the proof of 9.2, given \( C \in B_0 \), there exists \( b'_{ij} \in \mathcal{O}_C \) such that the Newton-Cramer polygon of

\[
\sum_{i+j \leq s} b'_{ij} X^i Y^j
\]

coincides with the Newton-Cramer polygon of \( K \). Moreover,

\[
f - \sum_{i+j \leq s} b'_{ij} X^i Y^j \in m_C^{s+1}
\]

and \( b'_{ij} \neq 0 \) if \( b_{ij} \neq 0 \). Now the transversality condition implies that \( \bar{X}, \bar{Y} \) generate the maximal ideal \( \bar{m}_C \) of \( \mathcal{O}_{S,C} \), and so we have

\[
f - \sum_{i+j \leq s} b_{ij} \bar{X}^i \bar{Y}^j \in \bar{m}_C^{s+1}
\]

where for an element \( h \in \mathcal{O}_C \), \( \bar{h} \) denotes its restriction to \( S \). From this relation the last part of the statement follows immediately. \( \square \)
10.2. Corollary

The local characteristic numbers of degeneration free conditions are additive.

Proof

Lemma 10.1 allows us to deal with locally plane curves instead of conditions (not necessarily reduced). Now for locally plane curves it is well known (cf. [Wal]) that the negative slopes of the sides of the Newton-Cramer polygon are the first exponents in the Puiseux series of the branches of the curve and that the sum of the orders of the branches corresponding to a given side is the number of integral points on that side decreased by 1 from what the additivity for curves is clear.

To describe the particular surface $S$ which will be used, fix a triangle $T$ in $\mathbb{P}_2$ and set $x_i, u_i$, $i=0,1,2$, to denote its vertexes and sides, respectively, with $u_i$ the side opposite $x_i$. Let $S=S_T$ be the surface in $\mathbb{W}$ of all conics for which $T$ is a self-polar triangle. If we take $T$ as a projective system of coordinates then the conics $C=(c,c')$ of $S_T$ are precisely those for which the matrices of $c$ and $c'$ are diagonal.

Let us denote by $u_i^2$ the point conic consisting of the line $u_i$ counted twice and by $x_i^2$ the envelope consisting of the pencil of lines through $x_i$ counted twice. Then $p(S_T)$ is a plane in $\mathbb{P}_5$ and $p:S_T \rightarrow p(S_T)$ is the blowing up of $p(S_T)$ at the points $u_o^2$, $u_1^2$, and $u_2^2$. Similarly, $t(S_T)$ is a plane in $\mathbb{P}_5$ and $t:S_T \rightarrow t(S_T)$ is the blowing up of $t(S_T)$ at the points $x_o^2$, $x_1^2$, and $x_2^2$. Equivalently, $S_T$ is the graph of the quadratic Cremona transformation $p(S_T) \rightarrow t(S_T)$ whose fundamental triangles are $(u_o^2,u_1^2,u_2^2)$ in $p(S_T)$ and $(x_o^2,x_1^2,x_2^2)$ in $t(S_T)$.

The intersection $S_T \cap A$ consists of the three mutually disjoint pencils $H_i$, $i=0,1,2$, defined as follows: $H_i$ is the pencil of pairs of lines that have $x_i$ as a double point and harmonically divide the pair of lines $(u_j,u_j)$.