A NOTE ON INTEGRABILITY OF $C^r$-NORMS OF
STOCHASTIC FLOWS AND APPLICATIONS

by

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Abstract

The integrability of uniform norms for higher derivatives of stochastic flows proved in this note implies existence of non-random stable manifolds, a Pesin type entropy formula and Hölder continuity of certain invariant subbundles for $C^2$-stochastic flows.

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1. Introduction

Let $\mathcal{M}$ be a compact smooth connected $m$-dimensional manifold and let $\text{Diff}^r\mathcal{M}$, $1 \leq r \leq \infty$, denote the group of $C^r$ diffeomorphisms of $\mathcal{M}$. A random process $\delta_t = \delta_t(\omega)$, $0 \leq t < \infty$ defined on some probability space $(\Omega, \mathcal{F}, P)$ with values in $\text{Diff}^r\mathcal{M}$ is called a stochastic flow if $\delta_0$ is the identity map id and

(i) the increments $\delta_{t_i}^{-1} \circ \delta_{t_{i-1}}^{-1}$, $i = 1, \ldots, n$, for any $t_0 \leq t_1 \leq \ldots \leq t_n$ are independent;

(ii) for $s \leq t$ the law of $\delta_s^{-1} \circ \delta_t$ depends only on $t - s$;

(iii) with probability one $\delta_t$ has continuous sample paths.

If $\nu_t$ is the distribution of $\delta_t$ then (i) and (ii) imply

$$\nu_{t} \ast \nu_{s} = \nu_{t+s} \quad (1.1)$$

i.e.

$$\int \phi(g) d\nu_{t+s} = \int \phi(g) d\nu_{t}(g) d\nu_{s}(f).$$

In fact, usually it suffices to assume (1.1) in place of (i) and (ii).
Let \( \| \delta \|_p \) denote the \( C^p \) norm of a diffeomorphism \( \delta \) (see [F]) which is defined by
\[
\| \delta \|_p = \sum_{i=0}^{r} \max_{j} \sup_{x \in U_j} \| D^i \delta (\delta^{-1}_j(x)) \|
\]
where \( U_j = \Phi_j V_j \) and \( \{ (V_j, \Phi_j), j = 1, \ldots, e \} \) is a system of charts.

Modifying an argument from [Bl] and [B2] we shall prove here the following result.

Proposition. If \( \delta_t \in \text{Diff}^p(M), 0 \leq t < \infty \) is a stochastic flow then
\[
E(\left( \sup_{0 \leq s \leq t} \| \delta_s \|_p \right)^k + (\sup_{0 \leq s \leq t} \| \delta_s^{-1} \|_p)^k) < \infty \tag{1.2}
\]
for any \( t > 0 \) and \( k = 1, 2, \ldots \) where \( E \) denotes the expectation on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

For \( r = k = 1 \) this result was proved in [B2]. The necessity of (1.2) for bigger \( r \), probably, needs some justification. It is known in dynamical systems that results connected with regularity properties of invariant subbundles in Oseledec's multiplicative ergodic theorem, stable manifolds' theorems and Pesin's type entropy formula require more than \( C^1 \) smoothness of corresponding diffeomorphisms. In case of stochastic flows these lead to certain integrability assumptions on \( C^2 \) norms (which usually can be weakened to the integrability of \( C^{1+\alpha}, \alpha > 0 \) norms). These applications of (1.2) will be discussed in Section 2. In Section 3 we shall prove (1.2).

By [Bl] all stochastic flows emerge essentially from stochastic differential equations. Still, the (1.2)-type integrability of uniform \( C^p \) norms can not be obtained directly from stochastic differential equations without using properties (i) - (iii) of the process \( \delta_t \) in \( \text{Diff}^p(M) \).

2. Applications of (1.2)

The following is a version of Oseledec's multiplicative ergodic theorem for stochastic flows.

Theorem. ([C], [K]) Let \( \rho \) be an invariant ergodic probability measure on \( M \) for a stochastic flow \( \delta_t \) i.e. \( E_\rho (\delta_t^{-1} \Gamma) = \rho (\Gamma) \) for any Borel \( \Gamma \subseteq M \). Then for \( \rho \times \mathbb{P} \) almost all \( (x, \omega) \) there exist a sequence of linear subspaces of the tangent