CHAPTER IV
STRONG LOCAL MINIMA

The purpose of this chapter is to prove a global maximum principle for optimal periodic control of functional differential equations. The proof is based on Ekeland's Variational Principle. For ordinary differential equations, the result reduces to Pontryagin's Maximum Principle (with periodic boundary conditions). Also a condition referring to optimal choice of the period is given.

The main results of this chapter are Theorem 2.1 and Theorem 2.2.

1. Problem Formulation

We consider the following optimal periodic control problem for retarded functional differential equations.

Problem 1.1 Minimize \( \frac{1}{(t_2-t_1)} \int_{t_1}^{t_2} g(x(t),u(t),t) dt \)

s.t. \( \dot{x}(t) = f(x_t,u(t),t), \) a.a. \( t \in T: = [t_1,t_2] \)

\( x_{t_1} = x_{t_2} \)

\( u(t) \in \Omega, \) a.a. \( t \in T, \)

where \( \Omega_x \subset \mathbb{R}^n, \) \( \Omega_u \subset \mathbb{R}^m, \) and \( \Omega_{\varphi} \subset C(-r,0;\mathbb{R}^n) \) are open sets, \( g: \Omega_x \times \Omega_u \times T \to \mathbb{R}, \) \( f: \Omega_x \times \Omega_u \times T \to \mathbb{R}^n, \) and \( \Omega \subset \Omega_u. \) (Here the indices in \( \Omega_x, \Omega_u, \) etc. refer to the variables chosen in these sets.)

In this problem, controls \( u \) in the following set of admissible controls are allowed:

\[ U_{ad} = \{ u \in L^\infty(T;\mathbb{R}^m): u(t) \in \Omega \ a.e. \}. \]  \hspace{1cm} (1.1)

Observe that

\[ U_{ad} \subset \partial U = \{ u \in L^\infty(T;\mathbb{R}^m): u(t) \in \partial \Omega \ a.e. \}. \]

We will prove a necessary optimality condition for pairs \( (x^0,u^0) \) which are optimal in the following sense.
Definition 1.2 Let \( x^0 \in C(t_1-t, t_2; R^n) \) and \( u^0 \in L^\infty(t_1, t_2; R^m) \) satisfy the constraints of Problem 1.1. The pair \((x^0, u^0)\) is called a strong local minimum of Problem 1.1 if there is \( \varepsilon > 0 \) such that for all pairs \((x, u)\) satisfying the constraints of Problem 1.1 with \(|x-x^0| < \varepsilon\) the following inequality holds:

\[
\frac{1}{(t_2-t_1)} \int_{t_1}^{t_2} g(x^0(t), u^0(t), t) \, dt \leq \frac{1}{(t_2-t_1)} \int_{t_1}^{t_2} g(x(t), u(t), t) \, dt \quad (1.2)
\]

Strong local minima are global minima with respect to \( u \).

Henceforth \((x^0, u^0)\) will denote a strong local minimum of Problem 1.1 and \( \phi^0 := x^0 \in 0^x \).

Extend \( f, g, x^0, \) and \( u^0 \) in a periodic way to the whole real line. Then \( x^0, u^0 \) satisfy the system equation for a.a. \( t \in R \).

Throughout this chapter, we assume, mostly without further mentioning, that the following hypotheses are satisfied.

Hypothesis 1.3 The functions \( g(x, u, t) \) and \( f(\varphi, u, t) \) are measurable in \( t \) and jointly continuous in \((x, u)\) and \((\varphi, u)\), respectively, and continuously Fréchet differentiable in \( x \) and \( \varphi \), respectively.

Hypothesis 1.4 There exists a monotonically increasing function \( q: R_+ \rightarrow R_+ \) such that for all \( x \in 0^x \), \( \varphi \in 0^\varphi \), \( \omega \in \Omega \) and a.a. \( t \in T \):

\[
|g(x, \omega, t)| + |g_x(x, \omega, t)| \leq q(|x|) \\
|f(\varphi, \omega, t)| + |D_1 f(\varphi, \omega, t)| \leq q(|\varphi|).
\]

Hypothesis 1.5 For every \( u \in U_{ad} \) the initial value problem

\[
x_{t_1} = \varphi^0, \quad \dot{x}(t) = f(x(t), u(t), t) \quad \text{a.a. } t \in T \quad (1.3)
\]

has a solution \( x(u, \varphi^0) \) on \( T \) and \( x(u, \varphi^0) \in 0^x \) is uniformly bounded for \( u \in U_{ad} \), say

\[
|x(u, \varphi^0)| \leq c_0, \text{ for all } t \in T.
\]

For the linearized system, we assume that the fundamental extendability property (III.2.2) holds:

Hypothesis 1.6 For all \( x^1 \) and \( u^1 \) with \(|x^1-x^0|_\infty \) and \(|u^1-u^0|_2 \) small enough there exists \( k \in L^2(t_1-r, t_2; R) \) such that