Four Lectures on the Differentiable Approach to General Equilibrium Theory

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With few exception the material from the first three lectures is taken from A. Mas-Colell: The Theory of General Economic Equilibrium: A Differentiable Approach, Cambridge University Press, 1985. We refer to this text for many extensions and the basic references. The names of the developers of the differentiable approach (at least for the parts covered in these lectures) should, however, be mentioned at the outset: G. Debreu, S. Smale, E. Dierker, and Y. Balasko.

The fourth lecture gives an account of a recent and fascinating development. A major and deep application of the differentiable approach to an area, incomplete market theory, not covered by the above reference.

Lecture I: Single Consumer Theory

1.1 Preference and Utility

The consumers making an appearance in these lectures have preferences defined over nonnegative vectors of $\mathbb{R}^\ell$, $\ell$ being the number of commodities. The consumption set is thus $\mathbb{R}^\ell_+$. A preference relation $\succeq$ is a relation $\succeq \subset \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$ with the properties:

(i) $x \succeq x$ for all $x \in \mathbb{R}^\ell_+$ (reflexivity).
(ii) "$x \succeq y$ and $y \succeq z$" $\Rightarrow$ "$x \succeq z$" (transitivity)
(iii) for every $x,y$ we have that either $x \succeq y$ on $y \succeq x$ (completeness).

In addition we always assume that $\succeq$ satisfies a topological property (which does not belong to the essence of the concept of preferences).

(iv) $\succeq$ is a closed set (continuity).

By a classic theorem (due to Eilenberg and Debreu) every relation $\succeq$ satisfying (i)–(iv) is representable by a utility function, i.e., there is a $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that "$x \succeq y$" $\iff$ "$u(x) \geq u(y)$". Moreover $u$ can be taken to be continuous. Of course, $u$ is not unique. What is intrinsic to $\succeq$ are the family of level curves of $u$ (called indifference sets), not the particular indexing (see Figure 1.1):

We read $x \succeq y$ as "at least as good", if $x \succeq y$ does hold but $y \succeq x$ does not (resp. does) then we say that $x$ is preferred to $y$ (resp., is indifferent to $x$), denoted $x \succ y$ (resp., $x \sim y$).

1.2 Properties of Preferences

(i) Monotonicity: A $\succeq$ is monotone (resp., strictly monotone) if "$x \succeq y$" $\Rightarrow$ "$x \succeq y$" (resp., $x \succeq y, x \neq y$ $\Rightarrow$ "$x \succ y$"). That is, commodities are not noxious (resp., they are desirable). See Figure 1.2.
(ii) **Boundary Condition**: Given $\succeq$, for every $x \gg 0$ the at least as good set $\{y: y \succeq x\}$ is closed relative to $\mathbb{R}^k$, i.e., every commodity is indispensable. See Figure I.3.

**N.B.**: Unless otherwise stated we assume from now on that preferences satisfy the strict monotonicity and the boundary conditions.

It is to be emphasized that these restrictions are not essential to the theory. They simply allow for ease of presentation. In particular, the boundary condition allows us to regard $R_{++}^k = \{x \in \mathbb{R}^k: x \gg 0\}$ as the consumption set.

(iii) **Convexity**: A $\succeq$ is **convex** (resp., strictly convex) if $\{y: y \succeq x\}$ is a convex set for every $y$ (resp., $\alpha y + (1 - \alpha)x \succeq x$ whenever $y \succeq x$ and $0 \leq \alpha < 1$). See Figure I.4.

If $\succeq$ is generated from a concave (resp., strictly concave) utility then $\succeq$ is convex (resp., strictly convex). The converse need not hold (i.e., there are convex preferences not generated by concave utilities).