Height pairing between algebraic cycles

A.A. Beilinson

Introduction

Let X be a smooth projective variety over Q; assume that its L-functions $L(H^j(X), s)$ satisfy the standard analytic continuation conjectures. Let $CH^i(X)$ be the group of codimension i cycles on X modulo rational equivalence, and $Ch^i(X)^o \subset CH^i(X)$ be the subgroup of cycles homologous to zero on $X(\mathbb{C})$; in particular $CH^1(X)^\circ = \mathrm{Pic}^\circ(X)(\mathbb{Q})$. The conjecture of Birch and Swinnerton-Dyer claims that at $s = 1$ the function $L(H^1(X), s)$ has zero of order rk $CH^1(X)^\circ$ with the leading coefficient equal to the determinant of Neron-Tate canonical height pairing multiplied by the period matrix determinant up to some rational multiple (we do not need its exact value in what follows). As for the other L-functions, Swinnerton-Dyer conjectured [20] that the function $L(H^{2i-1}(X), s)$ has at s=i (=the middle of the critical strip) the zero of order rk $CH^i(X)^\circ$. The aim of this note is to define the canonical height pairing between $CH^i(X)^\circ$ and $CH^{\dim X+1-i}(X)^\circ$ that coincides with the Neron-Tate one for i=1 and whose determinant multiplied by the period matrix determinant should conjecturally be equal up to a rational multiple (of the nature I cannot imagine) to the leading coefficient of $L(H^{2i-1}(X), s)$ at $s=i$*). This pairing should also occure in Riemann-Roch type theorems a la Arakelov-Faltings (see [15]).

---

*) In fact our height pairing is defined on a certain subgroup of $CH^{\cdot}(X)^\circ$; under the very plausible (:=of rank of evidence far higher then B-SwD) local conjectures this subgroups should coincide with the whole $CH^{\cdot}(X)^\circ$.

The paper goes as follows. To motivate the basic construction, we begin with the simpler geometric case: here our base field is a field $k(C)$ of rational functions on a smooth projective curve C. Then the height pairing $\langle , \rangle$ comes from the global Poincaré duality on ℓ-adic cohomology. We may compute $\langle , \rangle$ in terms of local data round the points of C : if a_1, a_2 are cycles with disjoint supports that are homologous to zero on $X \otimes \overline{k(C)}$ and $v \in C$ is a closed point then the local link index $\langle a_1, a_2 \rangle_v$ is defined, and we have

$$(*) \qquad \langle a_1, a_2 \rangle = \sum_{v \in C} \langle a_1, a_2 \rangle_v$$

In the arithmetic situation, when the base field is a number field, the global construction fails due to the lack of appropriate cohomology theory. But we may still define the local indices $\langle , \rangle_v$ numbered by the places of the base field, and then use $(*)$ as the definition of $\langle , \rangle$. These indices are defined using ℓ-adic cohomology for non-archimedean v and using the absolute Hodge-Deligne cohomology (see [2], [3], [19]) for archimedean ones; in case of pairing between divisors and zero cycles they are just Neron's quasifunctions ([17], [13], [22])

We also consider the intersection pairings. In the geometric case this is just the usual intersection pairing between the cycles of complementary dimensions on the regular scheme X_C proper over C. In the arithmetic case the role of X_C plays the A-variety $X = (X_{\mathbb{Z}}, \omega)$: where $X_{\mathbb{Z}}$ is a regular scheme projective over Spec $\mathbb{Z}$ and ω is a Kahler (1,1)-form on $X_{\mathbb{Z}} \otimes \mathbb{R}$ (see [15]). We define the corresponding Chow groups $CH^{\cdot}(X)$ and the $\mathbb{R}$-valued intersection pairing between $CH^i(X)$ and $CH^{\dim X - i}(X)$. This construction was independently found by H. Gillet and Ch. Soulé [10].

The final § contains some conjectures and motivic speculations about algebraic cycles, heights, L-functions and absolute cohomology groups.

The different construction of height pairing was proposed by S. Bloch [6]; I hope that our pairings coincide.

I would like to thank S. Bloch, P. Deligne, Yu. Manin, V. Schechtman and Ch. Soulé for stimulating ideas and interest.