LARGE TRANSCENDENCE DEGREE REVISITED
I. EXPONENTIAL AND NON-CM CASES

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I. Introduction.

The first results on the algebraic independence of more than one number using elimination theory were obtained by A.O. Gelfond (see [Br1] for an account of the method and [Wa2] for an update with complete bibliography). He employed elementary properties of resultants to establish a criterion for algebraicity involving the values of a sequence of integral polynomials with slowly growing size. In order to establish the algebraic independence of at least two numbers out of rather small sets of numbers related by the exponential function, he also proved a bound for the number of zeros an exponential polynomial can have in a disk of radius \( R > 0 \). G.V. Chudnovsky [Ch] was the first to give an extension of Gelfond's method, based on the successive use of his semi-resultants, for showing the algebraic independence of arbitrarily many numbers out of (exponentially larger) sets of numbers related by the exponential function. For this he also required the "small values theorem" of R. Tijdeman [Tij], which was a stronger quantitative form of Gelfond's zero estimate. Further work on this method was carried out by P. Warkentin [War], R. Endell [En], E. Reyssat [Re] and P. Philippon [Phi], and Yu.V. Nesterenko [Ne3].

Through several fundamental advances, D.W. Masser and G. Wüstholz [Ma-WU2] were able to establish the analogues of Chudnovsky's statements for Weierstrass elliptic functions without complex multiplications. They gave explicit bounds for the degrees in the Hilbert Nullstellensatz to provide a replacement for the elimination techniques via (semi-) resultants. They also used commutative algebra to refine their basic algebraic zero estimate of [Ma-WU1]. They developed an effective elliptic version of a theorem of E. Kolchin on subgroups of products of algebraic groups and applied it to guarantee that the polynomials used to express the values generated by the auxiliary function satisfy the conditions of Hilbert's Nullstellensatz. It became clear that an optimal improvement of the bounds in the degrees appearing in the Nullstellensatz would reduce the cardinality of the sets of numbers containing at least \( k \) algebraically independent members to \( O(k^2) \). Such bounds have now been established in [Br2] using the powerful tools of Nesterenko [Ne1]-[Ne3] and Philippon [Ph4] for elimination and a generous suggestion of C. Berenstein and A. Yger to employ deep results from the theory of several complex variables.

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In the meantime, Philippon has obtained a version of the zero estimates for general commutative algebraic groups \([\text{Ph5}]\), which is best possible in several respects. In addition he has obtained a generalization \([\text{Ph2}], [\text{Ph4}]\) of Gelfond's criterion, now involving a sequence \(I_N\) of polynomial ideals over an algebraic number field, whose generators are small at a fixed point \(\omega\) of \(\mathbb{C}^n\) or \(\mathbb{C}_p^n\), where each \(I_N\) has only finitely many zeros inside a ball about \(\omega\) of radius \(\rho_N > 0\). These results are of sufficient strength to yield the algebraic independence of \(k\) numbers out of specific sets of \(O(k)\) numbers related by either the exponential function or a Weierstrass elliptic function without complex multiplication. Thus the consequences of Philippon's result are even stronger than those provided in the standard way \([\text{Ma-Wt2}]\) by the sharp Nullstellensatz.

Quite recently the author has used the tools developed by Nesterenko and Philippon to establish sharp lower bounds on the maximum absolute values of integral polynomials having no common zero within a ball of radius \(\rho > 0\) centered at a fixed point in \(\mathbb{C}^n\) \([\text{Br3}]\). This allows us to relax a technical hypothesis in \([\text{Ph4}]\) from a measure of linear independence to an intermittent lower bound which must hold "only" for infinitely many values of the parameter. The corresponding remark also applies to the principal results of \([\text{Wa3}]\). Our proof resembles that of \([\text{Ma-Wt2}]\) insofar as no sequence of ideals or criterion for algebraic independence is invoked. However it resembles that of \([\text{Ph4}]\) insofar as we concern ourselves with the question of possible zeros of the ideal only near \(\omega\). Moreover we obtain in a natural manner extensions of many of the quantitative applications of \([\text{Ph3}], [\text{Ne4}], [\text{Ne5}], [\text{Ja}]\).

It is the purpose of this paper to carry out these applications and to indicate briefly how the two methods compare. If all the numbers involved in any one of the theorems could be shown to be algebraically independent, then the corresponding quantitative result would be a lower bound for any non-zero polynomial over \(\mathbb{Z}\) in these numbers. Present independence results using these methods are unfortunately not so strong. Therefore we recall the definitions of the appropriate quantitative analogues, first devised by Philippon \([\text{Ph3}]\).

Following that we briefly sketch the two elimination techniques. Then we derive the properties which are actually provided by the combination of Philippon's notion of redundant variables in the auxiliary functions, the Masser-Wüstholz effective version of Kolchin's theorem for elliptic functions without complex multiplications, and Philippon's zero estimates for algebraic groups. This is the raw material for both of the elimination techniques. Finally we deduce the results from the two approaches for algebraic independence. In the second part of this paper, which is joint work with R. Tubbs, we treat the case of elliptic functions with complex multiplications.

I am indebted to D.W. Masser and R. Tubbs for many helpful conversations. Apparently Tubbs was the first to prove zero estimates which correspond to zero-free regions \([\text{Tu}]\). As far as I am aware,