1. Introduction.

Subharmonic functions have been used in the study of minimal surfaces already by Rado [5] and Bechenbach and Rado [1]. The purpose of this paper is to study (pluri-)subharmonic functions having graphs with smallest positive area.

Let $U$ be a domain in $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $S$ be a weak*–closed set of subharmonic functions on $U$. If $b$ is a given continuous function on $\partial U$, the problem is to find $h \in S$ with $\lim_{z \to x, z \in U} h(z) = b(x)$, $\forall x \in \partial U$ and such that $h$ minimizes graph area among the functions in $S$.

2. Existence of solution.

Theorem 1. Let $U$ be a bounded subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $b$ be a continuous function on $\partial U$ and assume there is a convex function $\varphi$, which is the envelope of the affine functions it dominates, and a harmonic function $h$ such that $\lim_{z \to x, z \in U} \varphi(z) = \lim_{z \to x, z \in U} h(z) = b(x)$, $\forall x \in \partial U$.

Assume that $S$ is a weak*–closed family of subharmonic functions containing all the convex functions and stable under $\sup$. Then there is a function $U$ in $S$ which minimizes graph area among the functions in $S$.

Proof: Let $S_1$ denote the functions in $S$ that are minimized by $\varphi$ and majorized by $h$. Then $S_1$ is weak*–compact and we prove first that there is a $v \in S_1$ that minimizes $A(v) = \sqrt{1 + |\text{grad } v|^2}$. Put $\alpha = \inf_{v \in S_1} A(v)$ and choose $v_j \in S_1$, such that $A(v_j) \to \alpha$ as $j \to +\infty$.

Observe that since $v_j \in S_1 \cap L^{1}\text{loc}(U) |\text{grad } U_j|^2 \in L^{1}\text{loc}(U)$ and we can assume that $v_j \to v \in S_1$ (otherwise, pick a subsequence). We claim that $A(v) = \alpha$.

Now $A(v) \geq \alpha$ by the definition of $\alpha$. Let $U'$ be a relatively compact subset of $U$. We can then select convex combinations $w_k = \sum_j \theta_j^k v_j$ where $0 \leq \theta_j^k \leq 1$, $\theta_j^k = 0$, $j < k$ and $\sum_j \theta_j^k = 1$ such that $w_k$ tends strongly to $w$ in potential theoretic sense (cf. Landkof
Therefore,
\[ \left( \sqrt{1 + |\text{grad } v|^2} \right)^2 \rightarrow 0, \]
as \( k \rightarrow \infty \). But since \( \sqrt{1 + |x|^2} \) is convex we get
\[ \int_{U'} \sqrt{1 + |\text{grad } v|^2} = \lim_{n \to +\infty} \int_{U'} \sqrt{1 + |\text{grad } w_k|^2} \]
\[ \leq \lim_{k \to +\infty} \sum_{j} \int_{U'} \sqrt{1 + |\text{grad } v_j|^2} \leq \lim_{k \to +\infty} \int_{U'} \sqrt{1 + |\text{grad } v_k|^2} \leq a. \]
Thus \( \int_{U'} \sqrt{1 + |\text{grad } v|^2} \leq a \) for every relatively compact \( U' \) in \( U \) which proves the claim.

To prove the theorem, it is enough to prove that if \( v \in S \), \( \lim_{Z \to X} v(Z) = b(X) \), \( \forall X \in \partial U \) and
\[ \max(v, g - \frac{\varepsilon}{2}) = b(X) \]
then \( v \geq \varphi \).

To get a contradiction, assume that \( P_{\varepsilon} = \{ z \in U : v \leq \varphi - \varepsilon \} \neq \emptyset \)
for any \( \varepsilon > 0 \). Then there is a \( z_0 \in P_{\varepsilon} \) and an affine function \( g \)
such that \( g \leq \varphi \) on \( U \) with equality at \( z_0 \). Therefore \( v_\varepsilon = \max(v, g - \frac{\varepsilon}{2}) \in S \), \( \lim_{Z \to X} v_\varepsilon(Z) = b(X) \) and
\[ \int_{U} \sqrt{1 + |\text{grad } v_\varepsilon|^2} = \int_{v_\varepsilon = v} \sqrt{1 + |\text{grad } v_\varepsilon|^2} + \int_{v_\varepsilon < v} \sqrt{1 + |\text{grad } v_\varepsilon|^2} < \]
\[ \int_{U} \sqrt{1 + |\text{grad } v|^2} = a, \]
since
\[ \int_{v < v_\varepsilon} \sqrt{1 + |\text{grad } v_\varepsilon|^2} < \int_{v < v_\varepsilon} \sqrt{1 + |\text{grad } v|^2}, \]
which is a contradiction.