SOME PROPERTIES OF THE CANONICAL MAPPING OF A
COMPLEX SPACE INTO ITS SPECTRUM

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SUMMARY. Conditions on the cohomology and on the singular locus of a complex
space \( X \) are given for the canonical mapping of \( X \) into its spectrum being
surjective or a homeomorphism. Especially, the case of the unbounded dimension
is studied.

Let \((X, \mathcal{O})\) be a complex space and \( \xi : X \rightarrow \text{Sp} X \) be the canonical mapping of
\( X \) in its spectrum \( \text{Sp} X \). If \( \dim X < \infty \), it is known that \( \xi \) is a homeomorphism
iff \( X \) is Stein and that \( X \) is Stein iff \( X \) is holomorphically separable and
\( H^q(X, \mathcal{O}) \) has countable dimension as a vector space on \( C \) for every \( q \geq 1 \) (see \(|F|, |J|\)). When the dimension of \( X \) is unbounded, some results of Markoe \(|M|\) and
Ephraim \(|E|\) are known about the Michael's conjecture on \( X \); but, to our knowledge,
little more is known (e.g. see \(|H|\) pág.50).

In the first part of this paper some properties of the mapping \( \xi \) are given for
Stein spaces of unbounded dimension. The following result (see (1.6)) is proved
together with various of its improvements. Let \( X \) be a reduced holomorphically
spreadable complex space. Assume that the singular locus \( S(X) \) of \( X \) is a Stein
subspace which verifies Michael's conjecture (e.g. \( S(X) \) is a finite dimensional

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Then $X$ is a Stein space iff $\xi : X \to \text{Sp} X$ is a homeomorphism or iff $\dim_\mathbf{C} H^q(X, \mathcal{O})$ is countable for every $q \geq 1$. Later we shall see that if $X$ is a Stein space, then $\xi : X \to \text{Sp} X$ is a homeomorphism (equivalently: $X$ verifies Michael's conjecture) iff $S(X)\xi : S(X) \to \text{Sp} S(X)$ is a homeomorphism (equiv: $S(X)$ verifies Michael's conjecture).

In $\mathcal{H}$ some interesting characterizations of the surjectivity of $\xi$ onto the continuous spectrum are given in the finite dimensional case. Here, from another point of view we are interested in studying how cohomological conditions regarding the structure sheaf imply the surjectivity of $\xi$ and whether it is possible to use rather elementary methods.

Effectively in the last paragraphs we find some results improving $|\mathcal{M}|, |\mathcal{E}|$ without explicit Steinness assumptions. Although some techniques of $|\mathcal{B}|, |\mathcal{E}|$ and $|\mathcal{J}|$ are used, we do not employ the results of $|\mathcal{J}|$ or of $|\mathcal{E}N|$ in the main result which is the following one. Let $(X, \mathcal{O})$ be a holomorphically spreadable complex space; let $\dim_\mathbf{C} H^q(X, \mathcal{O})$ be countable for every $q \geq 1$. Let $L$ be a closed subspace of $X$ such that the connected components of $X \setminus L$ are finite dimensional open subspaces of $X$. Let $\phi \in \text{Sp} X$. If $h \in \mathcal{O}(X)$ exists s.t. $\phi(h) \neq 0$ and $h(y) = 0$ for every $y \in L$, then $\phi$ is a point character, i.e. $x \in X \setminus L$ exists s.t. $\phi(f) = f(x)$ for every $f \in \mathcal{O}(X)$. Some consequences about the surjectivity of $\xi$ are deduced from this result.

Some of the results here obtained were the subject of the expose $|\mathcal{C}|$.

§ 0. PRELIMINARIES

Complex spaces $(X, \mathcal{O})$ that we are going to consider, have countable Hausdorff topology; they may be nonreduced. A character $\phi$ of $(X, \mathcal{O})$ is a homomorphism $\phi : \mathcal{O}(X) \to \mathbf{C}$ of $\mathbf{C}$-algebras, $\phi(1) = 1$; we do not assume $\phi$