Quantization and Unitary Representations

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Part I: Prequantization

0. Introduction. 1. This paper is the first part of a two part paper dealing with the question of setting up a unified theory of unitary representations of connected Lie groups.

We have found that when the notion of what the physicists mean by quantizing a function is suitably generalized and made rigorous, one may develop a theory which goes a long way towards constructing all the irreducible unitary representations of a connected Lie group. In the compact case it encompasses the Borel-Weil theorem. Generalizing Kirillov's result on nilpotent groups, L. Auslander and I have shown that it yields all the irreducible unitary representations of a solvable group of type I. (Also a criterion for being of type I is simply expressed in terms of the theory.) For the semi-simple case, by results of Harish-Chandra and Schmid, it appears that enough representations are constructed this way to decompose the regular representation.

The theory is founded in differential geometry. A principal point is that the 2-form of a symplectic manifold under a certain condition (integrality condition) is the curvature of a line bundle with connection; that the Hilbert space involved (as far as this paper is concerned) is to be found among the sections of this line bundle and

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that operating on these sections is a Lie algebra (under Poisson bracket) of functions on the manifold. This assignment mapping functions to operators is quantization.

The extraction of the Hilbert space and the Lie algebra above requires the notion of polarizing the symplectic manifold. (In the classical quantum mechanical situation, this means e.g. isolating the q's from the even dimensional space of p's and q's. The notion, however, is broad enough so that even in this case it also yields the Bargmann-Segal-Fock representation of the Heisenberg Lie algebra on the $z = q + ip$ as well as the usual one on the q's.) We will deal with this in Part II. In Part I we consider only pre-quantization (see § 4.3).

A representation of a group will arise from a symplectic homogeneous space X when the corresponding Lie algebra of hamiltonian vector fields can be lifted to functions which are quantizable. One of the results (Theorem 5.4.1) is that this is the case when and only when X corresponds to or covers an orbit in the dual of the Lie algebra, justifying the idea of finding all the irreducible representations from these orbits (as Kirillov did in the nilpotent case). This fact also yields a generalization of Wang's theorem characterizing compact Kahler homogeneous spaces. Also (Corollary 1 to Theorem 5.7.1) generalizing results of Borel-Weil in the compact case, one has that the 2-form on the orbit, defined by a linear functional on the Lie algebra f, is integral if and only if f is the differential of a character on the isotropy group (which may be disconnected) at f.

Part I is devoted to the differential geometry foundations of the theory. For completeness we include with proofs basic facts