ON KLEIN - MASKIT COMBINATION THEOREMS

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In this paper we intend to discuss an extension of the well-known combination theorems that F. Klein and B. Maskit have proved for Kleinian groups. This theorems give sufficient conditions that the group $G$ generated by the union of two discontinuous groups $G_1, G_2$ to be discontinuous and allow to construct a fundamental domain for $G$ by means of two corresponding fundamental domains of the groups $G_1, G_2$. Such theorems can be very useful in a classification theory of the discontinuous groups acting on a fixed space. They also can be used to construct groups with some special properties.

First of all it is necessary to remind some definitions.

**Definition 1** Let $X$ be a locally compact, connected topological space and $G$ a group of homeomorphisms of $X$. We say that $G$ acts discontinuously on the open set $U \subseteq X$ if for every compact set $K \subseteq U$ the set $G_K = \{ T \mid T \in G, \ TK \cap K \neq \emptyset \}$ is finite. By $\Omega(G)$ we denote the maximal open set on which $G$ acts discontinuously. The group $G$ will be called discontinuous if $\Omega(G) \neq \emptyset$.

**Definition 2** Let $G$ be a discontinuous group acting on $X$ and $D$ an open subset of $\Omega(G)$. We say that $D$ is a fundamental domain for $G$ if the following two conditions are satisfied.
i) $T \mathcal{D} \cap \mathcal{D} = \emptyset$ for all $T \in G - I$

ii) $\bigcup_{T \in G} T \mathcal{D} = \mathcal{L}(G)$ ($\mathcal{L}$ being the closure of $\mathcal{D}$ in $\mathcal{L}(G)$)

**Definition 3** The one point compactification of $\mathbb{R}^n$ will be denoted by $\mathcal{M}(n)$ and will be called the Moebius space of dimension $n$.

**Definition 4** The group of the transformations of $\mathcal{M}(n)$ generated by reflections in the spheres and hyperplanes of $\mathbb{R}^n$ will be denoted by $\text{GM}(n)$ and will be called the Moebius group of dimension $n$.

**Definition 5** A sequence $(G, \mathcal{D}, H, \Delta, S)$ in which $G$ is a discontinuous subgroup of $\text{GM}(n)$, $H$ is a subgroup of $G$, $\mathcal{D}$ ($\Delta$) is a fundamental domain for $G$ ($H$) and $S$ is a compact hypersurface of $\mathcal{M}(n)$, will be called conglomerate if the following conditions are satisfied:

a) $S \subseteq \Delta$

b) $S$ is invariant under $H$

c) there exists a neighborhood $V$ of $S$ such that $\Delta \cap V \subseteq \mathcal{D}$.

**Definition 6** The sequence $(G, \mathcal{D}, H, \Delta, S, B)$ will be called a big conglomerate if $(G, \mathcal{D}, H, \Delta, S)$ is a conglomerate, $B$ is one of the two connected components of $\mathcal{M}(n) - S$, invariant under $H$ and the following conditions are satisfied:

a) $\Delta \cap (B \cup S) = \mathcal{D} \cap (B \cup S)$

b) $\mathcal{D} \cap (B \cup S) \neq \mathcal{D}$

c) $T(S) \cap S = \emptyset$ if $T \in G - H$.

The last two definitions were introduced by B. Maskit in the case of the Kleinian groups.

Now, if we remark that the group $\text{GM}(n)$ have the same geometrical nature as the group all conformal mappings of the compact complex plane onto itself, we can restate some results, concerning the Kleinian groups, for the discontinuous subgroups of $\text{GM}(n)$.