O. Basic Model Theory

Introduction. For the reader's convenience we summarize certain basic notions of model theory which are treated at length in [11, 36, 43]. Foremost among these is the notion of a first order sentence ($\S 1$).

The most important theorem reviewed in this chapter is the Compactness Theorem ($\S 1$). This is most efficiently exploited via saturated models ($\S 2$).

§1. First order languages. First order sentences.

Structures and language are treated carefully in [11, 36, 43]. We will review the salient points.

A mathematical structure $\mathcal{A}$ (also called a relational system or model) consists of a set $A$ on which various functions $f_i$ and relations $R_j$ are defined; in addition various elements $c_k$ of $S$ may be distinguished. Thus a structure is a 4-tuple:

$\mathcal{A} = \langle A; \{f_i\}, \{R_j\}, \{c_k\} \rangle$. For example an ordered abelian group $A$ will be equipped with a binary operation $+:A \times A \to A$, an ordering $<$ (a binary relation on $A$), and a distinguished element $0$ (the identity):

$\mathcal{A} = \langle A, (+), (<), (0) \rangle$.

If $\mathcal{A} = \langle A; \{f_i\}, \{R_j\}, \{c_k\} \rangle$ is a structure, where $f_i$ is a function of $m_i$ arguments and $R_j$ is a relation of $n_j$-tuples of elements of $A$, we call $m_i$ or $n_j$ the rank of $f_i$ or $R_j$. If $\mathcal{A}' = \langle A', \{f'_i\}, \{R'_j\}, \{c'_k\} \rangle$ is a second structure in which rank $f'_i = \text{rank } f_i$, rank $R'_j = \text{rank } R_j$ for all $i, j$, then a map $h:A \to A'$ will be called an injection (monomorphism) of $\mathcal{A}$ into $\mathcal{A}'$ if it respects the functions, relations, and distinguished elements of $\mathcal{A}$ and $\mathcal{A}'$. In more detail we assume:

1. $h:A \to A'$ is one-one
2. $h(f_i(a_1, \ldots, a_{m_i})) = f'_i(ha_1, \ldots, ha_{m_i})$
3. $R_{j}a_{1}, \ldots, a_{n_{j}}$ holds if $R_{j}ha_{1}, \ldots, ha_{n_{j}}$ holds.

4. $hc_k = c'_k$.

Surjective injections are called bijections or isomorphisms. As we will make comparatively little use of general homomorphisms, we do not pause to define them here.

We now turn from the mathematical structures to a consideration of formal statements about mathematical structures. Consider the following simple statements in the language of ordered abelian groups:

1. $\forall x \forall y (x+y = y+x)$,
2. $\forall x \exists y (y+y = x)$,
3. $\exists x (x>0 \forall y (y>0 \Rightarrow [y>x \text{ or } y=x]))$.

The first statement is of course true in all abelian groups, while the second and third statements may be either true or false, depending on the group in question.

In formulating statements 1-3, we made use of:

a. Variables $x, y$, varying over the elements of our domain $A$.

b. Quantifiers $\forall, \exists$ (for all, there exists).

c. Relation symbols $>, =$; every ordered abelian group is of course equipped with a definite ordering $>$, and in addition every set whatsoever carries a canonical equality relation $=$.

d. A function symbol $+:$ every ordered abelian group is equipped with a binary function $+$.

e. A constant $0$ denoting the distinguished elements

f. Logical connectives (and, or, implies). Other connectives one might use are $\neg, \Leftrightarrow$ (read "not," "iff").

Punctuation: [ ] ( ) etc.

Mathematical statements which can be written down using only the symbols listed above (variables, quantifiers, relation, function symbols, constants, logical connectives, punctuation) are called first-order statements (in the language of abelian groups). Thus statements 1-3