5. Feynman path integrals for the anharmonic oscillator.

By the anharmonic oscillator with $n$ degrees of freedom we shall understand the mechanical system in $\mathbb{R}^n$ with classical action integral of the form:

$$S_t = \frac{m}{2} \int_{0}^{t} \dot{y}(\tau)^2 d\tau - \frac{1}{2} \int_{0}^{t} y A^2 y d\tau - \int_{0}^{t} V(y(\tau)) d\tau,$$  \hspace{1cm} (5.1)

where $A^2$ is a strictly positive definite matrix in $\mathbb{R}^n$ and $\dot{y}(\tau) = \frac{dy}{d\tau}$ and $V(x)$ is a nice function which in the following shall be taken to be in the space $\mathcal{F}(\mathbb{R}^n)$ i.e.

$$V(x) = \int_{\mathbb{R}^n} e^{i\alpha \cdot x} \, d\mu(\alpha),$$ \hspace{1cm} (5.2)

where $\mu$ is a bounded complex measure. We shall of course also assume, for physical reasons, that $V$ is real. \(^1\)Let $\varphi(x) \in \mathcal{F}(\mathbb{R}^n)$ with

$$\varphi(x) = \int_{\mathbb{R}^n} e^{i\alpha \cdot x} \, dv(\alpha),$$ \hspace{1cm} (5.3)

then we shall give a meaning to the Feynman path integral

$$\sim e^{i\int_{0}^{t} \frac{1}{2} \dot{y}^2 d\tau - \frac{1}{2} y A^2 y d\tau - i \int_{0}^{t} V(y(\tau)) d\tau} \varphi(y(0)) dy,$$ \hspace{1cm} (5.4)

by using the integral defined in the previous section. For simplicity we shall assume in what follows that $m = \hbar = 1$.

In the previous section we only defined integrals over linear spaces, so we shall first have to transform the non homogeneous boundary condition $y(t) = x$ into a homogeneous one. This is easily done if there exists a solution $g(\tau)$ to the following boundary value problem on the interval $[0, t]$:
\[ \ddot{\beta} + A^2 \beta = 0, \quad \beta(t) = x, \quad \dot{\beta}(0) = 0. \]  \hspace{1cm} (5.5)

Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( A \). If we now assume that
\[ t \neq (k + \frac{1}{2}) \frac{\pi}{\lambda_1} \]  \hspace{1cm} (5.6)

for all \( k = 0, 1, \ldots \), and \( i = 1, \ldots, n \), then (5.5) has a unique solution given by
\[ \beta(t) = \frac{\cos A t}{\cos A \frac{t}{\lambda_1}} x. \]  \hspace{1cm} (5.7)

We then make formally the substitution \( \gamma = \gamma + \beta \) in (5.4) and get
\[ \int_0^t e^{- \frac{1}{2} \int 0 (\gamma + \beta)^2 \, dt} \left( \int_0^t e^{- \frac{1}{2} \int 0 A^2 (\gamma + \beta) \, dt} \int_0^t e^{- \int_0^t V(\gamma(\tau) + \beta(\tau)) \, d\tau} \int_0^t e^{- \int_0^t V(\gamma(\tau) + \beta(\tau)) \, d\tau} \phi(\gamma(0) + \beta(0)) \, d\gamma. \]
\[ \gamma(t) = 0 \]  \hspace{1cm} (5.8)

Now, due to (5.5), we have that, if \( \gamma(t) = 0 \),
\[ \int_0^t (\dot{\gamma} + \beta)^2 \, dt - \int_0^t (\gamma + \beta) A^2 (\gamma + \beta) \, dt = \int_0^t \dot{\gamma}^2 \, dt - \int_0^t \gamma A^2 \gamma \, dt + \beta(t) \dot{\beta}(t). \]  \hspace{1cm} (5.9)

Since \( \beta(t) = x \) and \( \dot{\beta}(t) = -A \tan A t x \), we may write (5.8) as
\[ \int_0^t e^{- \frac{1}{2} \int 0 (\dot{\gamma}^2 - \gamma A^2 \gamma) \, dt} \left( \int_0^t e^{- \int_0^t V(\gamma(\tau) + \beta(\tau)) \, d\tau} \phi(\gamma(0) + \beta(0)) \, d\gamma. \]
\[ \gamma(t) = 0 \]  \hspace{1cm} (5.10)

Hence we have transformed the boundary condition to a homogeneous one. Let now \( \mathcal{H}_o \) be the real separable Hilbert space of continuous functions \( \gamma \) from \([0, t]\) to \( \mathbb{R}^n \) such that \( \gamma(t) = 0 \) and \( |\gamma|^2 = \int_0^t \gamma^2 \, dt \) is finite. In \( \mathcal{H}_o \) the quadratic form
\[ \int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma) \, dt \]  is obviously bounded and therefore given by a