5. Feynman path integrals for the anharmonic oscillator.

By the anharmonic oscillator with $n$ degrees of freedom we shall understand the mechanical system in $\mathbb{R}^n$ with classical action integral of the form:

$$ S_t = \frac{m}{2} \int_0^t \dot{y}(\tau)^2 d\tau - \frac{1}{2} \int_0^t \gamma A^2 \gamma d\tau - \int_0^t V(y(\tau)) d\tau, \quad (5.1) $$

where $A^2$ is a strictly positive definite matrix in $\mathbb{R}^n$ and $\dot{y}(\tau) = \frac{dy}{d\tau}$ and $V(x)$ is a nice function which in the following shall be taken to be in the space $\mathcal{F}(\mathbb{R}^n)$ i.e.

$$ V(x) = \int_{\mathbb{R}^n} e^{i\alpha x} d\mu(\alpha), \quad (5.2) $$

where $\mu$ is a bounded complex measure. We shall of course also assume, for physical reasons, that $V$ is real. 1) Let $\varphi(x) \in \mathcal{F}(\mathbb{R}^n)$ with

$$ \varphi(x) = \int_{\mathbb{R}^n} e^{i\alpha x} d\nu(\alpha), \quad (5.3) $$

then we shall give a meaning to the Feynman path integral

$$ \sim e^{\frac{i}{2h} \int_0^t \dot{y}^2 d\tau - \frac{i}{2h} \int_0^t \gamma A^2 \gamma d\tau - \int_0^t V(y(\tau)) d\tau} \int_{\mathbb{R}^n} e^{i\gamma(0) \alpha} \varphi(y(0)) d\gamma, \quad (5.4) $$

$\gamma(t)=x$

by using the integral defined in the previous section. For simplicity we shall assume in what follows that $m = \hbar = 1$.

In the previous section we only defined integrals over linear spaces, so we shall first have to transform the non homogeneous boundary condition $\gamma(t) = x$ into a homogeneous one. This is easily done if there exists a solution $\beta(\tau)$ to the following boundary value problem on the interval $[0, t]$:
\[ \ddot{\beta} + A^2\beta = 0, \quad \beta(t) = x, \quad \dot{\beta}(0) = 0. \]  

Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( A \). If we now assume that

\[ t \neq (k + \frac{1}{2}) \frac{\pi}{\lambda_1} \]

for all \( k = 0, 1, \ldots \), and \( i = 1, \ldots, n \), then (5.5) has a unique solution given by

\[ \beta(t) = \frac{\cos At}{\cos \frac{At}{\lambda_1}} x. \]

We then make formally the substitution \( \gamma = \gamma + \beta \) in (5.4) and get

\[ \int_0^t \left( \frac{1}{2}(\dot{\gamma} + \dot{\beta})^2 \right) dt - \int_0^t \frac{1}{2}(\gamma + \beta)A^2(\gamma + \beta) dt - \int_0^t V(\gamma(t) + \beta(t)) dt = \int_0^t x \cdot \frac{d}{dt} \phi(\gamma(0) + \beta(0)) dy. \]

Now, due to (5.5), we have that, if \( \gamma(t) = 0 \),

\[ \int_0^t \left( \dot{\gamma} + \dot{\beta} \right)^2 dt - \int_0^t (\gamma + \beta)A^2(\gamma + \beta) dt = \int_0^t \dot{\gamma}^2 dt - \int_0^t \gamma A^2 \gamma dt + \beta(t)\dot{\beta}(t). \]

Since \( \beta(t) = x \) and \( \dot{\beta}(t) = -A tg At x \), we may write (5.8) as

\[ -\frac{1}{2} x A tg A x - \frac{1}{2} \int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma) dt - \int_0^t V(\gamma(t) + \beta(t)) dt = \int_0^t x \cdot \frac{d}{dt} \phi(\gamma(0) + \beta(0)) dy. \]

Hence we have transformed the boundary condition to a homogeneous one. Let now \( \mathcal{H}_0 \) be the real separable Hilbert space of continuous functions \( \gamma \) from \([0, t] \) to \( \mathbb{R}^n \) such that \( \gamma(t) = 0 \) and \( |\gamma|^2 = \int_0^t \gamma^2 dt \) is finite. In \( \mathcal{H}_0 \) the quadratic form

\[ \int_0^t (\dot{\gamma}^2 - \gamma A^2 \gamma) dt \]

is obviously bounded and therefore given by a