Let \( R = \mathbb{R}[x, y]/(x^2 + y^2 - 1) \). Regarding \( R \) as a ring of functions on the circle gives a map from \( \text{SL}(n, R) \) to the function space \( \text{SL}(n, \mathbb{R}) S^1 \) and thus a map \( \text{SL}(n, R) \to \prod_1 \text{SL}(n, \mathbb{R}) \). These maps are compatible and give a map \( \text{SK}_1(R) \to \prod_1 \text{SL}(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z} \).

The matrix \( \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \) maps into a generator of \( \prod_1 \text{SL}(\mathbb{R}) \) so \( \text{SK}_1(R) \neq 0 \).

It can be shown that \( \text{SK}_1(R) = \mathbb{Z}/2\mathbb{Z} \).

Chapter 14. \( K_2(R) \)

In this chapter we present Milnor's definition of \( K_2(R) \).

Let \( R \) be a ring. Then \( E(R) \) is generated by \( 1 + ae_{ij} \) where \( a \in R \) and \( i \neq j \). These generators have relations

\[
(I + re_{ij})(I + se_{ij}) = (I + (r + s)e_{ij})
\]

and

\[
[I + se_{ij}, I + te_{k\ell}] = \begin{cases}
I + ste_{i\ell} & \text{if } j = k \text{ and } i \neq \ell \\
I & \text{if } j \neq k \text{ and } i \neq \ell \\
I - tse_{kj} & \text{if } j \neq k \text{ and } i = \ell.
\end{cases}
\]

Remarks. These relations are a special case of those studied by Chevalley and Steinberg for algebraic groups over fields.

Definition. Let \( R \) be a ring. Then the Steinberg group of \( R \), \( \text{ST}(R) \), is the (non abelian) group with generators \( x_{ij}(t) \) for all positive integers \( i \neq j \) and all \( t \in R \), and with the relations

\[
x_{ij}(s)x_{ij}(t) = x_{ij}(s + t)
\]

and

\[
[x_{ij}(s), x_{k\ell}(t)] = \begin{cases}
1 & \text{if } i \neq \ell \text{ and } j \neq k \\
x_{i\ell} & \text{if } j = k \text{ and } i \neq \ell.
\end{cases}
\]
Remarks. Since in any group \([a, b]^{-1} = [b, a]\), the relation 
\([x_{ij}(s), x_{ki}(t)] = x(-ts)_{kj}\) is an immediate consequence. The function \(f: \text{ST}(R) \rightarrow \text{E}(R)\) given by \(f(x_{ij}(t)) = I + e_{ij}\) is a homomorphism of groups. It is clear that \(\text{ST}\) is a functor from rings to groups. We could have defined \(\text{ST}(n, R)\), but it is not known if the map \(\text{ST}(n, R) \rightarrow \text{ST}(R)\) is a mono.

Definition. \(K_2(R)\) is the kernel of \(f: \text{ST}(R) \rightarrow \text{E}(R)\).

Remark. We have an exact sequence of functors
\[0 \rightarrow K_2 \rightarrow \text{ST} \rightarrow \text{GL} \rightarrow K_1 \rightarrow 0.\]

Theorem 14.1. \(K_2(R)\) is the center of \(\text{ST}(R)\).

Proof. \(Z(\text{ST}(R)) \subseteq K_2(R)\). Pick a \(e \in Z(\text{ST}(R))\). Since \(f\) is onto, \(f(e) \in Z(\text{E}(R))\). Hence it is enough to show that \(\text{E}(R)\) has a trivial center. We will show that \(C_{\text{GL}(R)}(\text{E}(R))\), the centralizer of \(\text{E}(R)\) in \(\text{GL}(R)\), is the identity. Suppose
\[
(\sum c_{ij} e_{ij})(1 + te_{uv}) = (1 + te_{uv})(\sum c_{ij} e_{ij}).
\]
Then \((\sum c_{ij} e_{ij})(te_{uv}) = (te_{uv})(\sum c_{ij} e_{ij})\). Therefore,
\[
\sum_{i} c_{iu} t e_{iv} = \sum_{j} t c_{vj} e_{uj}.
\]
Therefore, if \(i \neq u\), then \(c_{iu} t = 0\), and if \(v \neq j\), then \(t c_{vj} = 0\). But this happens for all \(t \in R\). Therefore, \(c_{ij} = 0\) if \(i \neq j\). \(c_{uu} t e_{uv} = t c_{vv} e_{uv}\) implies \(c_{uu} t = t c_{vv}\). Therefore, \(c_{uu} = c_{vv}\) and \((c_{ij}) = c I\) where \(c\) is in the center of \(R\). But \((c_{ij}) \in \text{GL}(R)\). Therefore \((c_{ij})\) is eventually the identity matrix. Hence \(c_{ii} = 1\) for all \(i\) and \((c_{ij})\) is the identity matrix. Therefore, \(Z(\text{ST}(R)) \subseteq K_2(R)\).

Now to show \(Z(\text{ST}(R)) \supseteq K_2(R)\). Let \(C_n\) be the subgroup of \(\text{ST}(R)\) which is generated by all \(x_{in}(t)\) where \(i \neq n\) and \(t \in R\). Let \(R_n\) be