THE HOMOLOGY OF $\mathcal{C}_{n+1}$-SPACES, $n \geq 0$

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The construction of homology operations defined for the homology of finite loop spaces parallels the construction of homology operations defined for the homology of infinite loop spaces, with some major differences. To recall, the operations for infinite loop spaces are defined via classes in the homology of the symmetric group, Σ_j. Working at the prime 2, Browder [24], generalizing and extending the operations of Araki and Kudo [1], found that an appropriate skeleton of $B\Sigma_2$ may be used to describe natural operations which allow computation of $H_*(\Omega^{n+1}\Sigma^{n+1}X;\mathbb{Z}_2)$ as an algebra. Dyer and Lashof [8], using similar methods, subsequently obtained partial analogous results at odd primes. However, comparison of the results of Dyer and Lashof to Milgram's computation [20] of $H_*(\Omega^{n+1}\Sigma^{n+1}X)$ as an algebra made it apparent that the skeleton of $B\Sigma_p$ intrinsic to the geometry of the finite loop space failed to give sufficient operations to compute $H_*(\Omega^{n+1}\Sigma^{n+1}X;\mathbb{Z}_p)$. To be precise, only $1/p-1$ times the requisite number of operations (defined in this paper) may be described using the methods of Dyer and Lashof.

In addition, there is a non-trivial unstable operation in two variables, λ_n, which was invented by Browder; the method of using finite skeleta of $B\Sigma_p$ does not lend itself to finding the relationships between λ_n and the other operations. These relationships are especially important in determination of fine structure and our later work in IV.

An alternative method for defining operations seemed to be provided by the composition pairing and the possibility that $H_*(\Omega^{n+1}S^{n+1};\mathbb{Z}_p)$ is universal for Dyer-Lashof operations. If $n = \infty$, this method works in principle, but it fails almost completely if $n < \infty$: It is shown in IV §6 that there are finite loop spaces with many non-trivial Dyer-Lashof operations for which the composition pairing is trivial.

The observation of Boardman and Vogt that the space of little (n+1)-cubes acts on (n+1)-fold loop spaces, together with May's theory of iterated loop spaces [G], led one to expect that the equivariant homology of the little cubes ought to enable one to define all requisite homology operations in a natural setting analogous to that provided for an infinite loop spaces by $B\Sigma_p$. This is the case. In addition, one can describe easily understood constructions with the little cubes which, when linked with May's theory of operads, enable one to determine the commutation relations between all of the operations, and between the operations and the product, coproduct, and Steenrod operations on the homology of iterated loop spaces.

Knowledge of this fine structure is essential, for example, in the analysis of the composition pairing and the Pontrjagin ring $H_*(SF(n+1);\mathbb{Z}_p)$ for all n and p in IV. Indeed, all of the formulas in III. 1.1-1.5 are explicitly used there. A further application of the fine structure is an improvement [28] of Snaith's stable decomposition for $\Omega^{n+1}\Sigma^{n+1}X$ [25].

We have tried to parallel the format of I as closely as possible, pointing out essential differences. Sections 1-4, which are analogous to I. 1,2,4, and 5, contain the computations of $H_*\Omega^{n+1}\Sigma^{n+1}X$ and $H_*C_{n+1}X$, $n > 0$, together with a catalogue of the relations amongst the operations.

In more detail, Section 1 gives a list of the commutation relations between all of the operations, coproduct, product, and between them and the Steenrod operations, conjugation, and homology suspension. The relationship between Whitehead products and the λ_n is also described.

Section 2 contains the definition of certain algebraic structures naturally suggested by the preceding section; the free versions of these algebraic structures are constructed.