Let $V$ be a vector space of finite dimension $n$ over an algebraically closed field $k$. The bilinear maps $V \times V \to V$ form a vector space $\text{Hom}_k(V \otimes V, V)$ of dimension $n^3$, which we do consider here together with its natural structure of an algebraic variety over $k$. Clearly, the associative maps form a Zariski-closed subset $S$ of $\text{Hom}_k(V \otimes V, V)$. A further simple investigation shows that the associative algebra-structures with 1 on $V$ form a Zariski-open affine subset of $S$, which we denote by $\text{Alg}_V$ or $\text{Alg}_n$. The purpose of this paper is to show that the algebra-structures (with 1) of finite representation type also form an open subset of $\text{Alg}_n$. In other words, if $A$ is an associative algebra with 1, $A$ is of finite representation type if some polynomials $P_1, \ldots, P_r$ do not vanish on the constant structures of $A$. This answers to a question of M. Auslander.

In §2 we write down explicitly some routine properties of the algebraic set $\text{Alg}_n$. Similar properties of the varieties of modules will be needed in §3. They are given in §1. In §3 we show that, for any natural number $r$, the algebra structures on $V$, for which there are only finitely many isomorphism classes of modules of dimension $\leq r$, form an open subset $S_r$ of $\text{Alg}_n$. Our main statement then follows from the fact that, for large $r$, $S_r$ is equal to the set of all algebra structures of finite representation type. This follows from the conjecture-theorem of Brauer-Thrall-Nazarova-Roiter: The last paragraph is devoted to a thorough investigation of $\text{Alg}_n$ for $n = 4$.

We denote schemes by "script letters" as $\mathbb{X}$. The corresponding roman letter $X$ then stands for the set $\mathbb{X}(k)$ of rational points of $\mathbb{X}$. 
§1. The module varieties.

1.1. Let $A$ be an associative algebra with 1 over the algebraically closed field $k$. For the sake of simplicity we suppose $A$ to be of finite dimension $n$ over $k$. If $W$ is a $k$-vector space of dimension $r$, the bilinear maps $AxW + W$ form a vector space $\text{Hom}_k(A \otimes W, W)$ of a dimension $r^2n$, whereas the $A$-module structures on $W$ form a Zariski-closed subset, which we denote by $\text{Mod}_A$. Clearly $\text{Mod}_A$ is the set of rational points of an algebraic scheme $\text{Mod}_A$ over $k$, which may be described in the functorial point of view as follows: for any commutative $k$-algebra $R$, we have

$$\text{Mod}_A(R) = \{R \otimes_k A\text{-module structures on } R \otimes_k W\}$$

The consideration of the scheme $\text{Mod}_A$ does not need any special commentary. For previous use of it we refer to Artin [1], Procesi [9] or Voigt [10] and recall some of their observations. The linear group $\text{GL}(W)$ obviously operates on $\text{Mod}_A$ by means of the formula $$(gf)w = g(f(a_0w), \text{ where } g \in \text{GL}(W), f \in \text{Hom}_k(A \otimes W, W).$$

The orbits of $\text{GL}(W) = \text{GL}(W)(k)$ in $\text{Mod}_A = \text{Mod}_A(k)$ are the isomorphism classes of $A$-modules with $k$-dimension $r$. If $M \in \text{Mod}_A$ is an $A$-module with underlying vector space $W$, we denote by $T_M$ the Zariski-tangent space of $\text{Mod}_A$ at the point $M$, by $T^0_M$ the Zariski-tangent space at $M$ of the orbit of $\text{GL}(W)$ through $M$. We then have the following Proposition (Voigt): For any $A$-module $M$ with underlying vector space $W$, there is a canonical isomorphism

$$T_M/T^0_M \cong \text{Ext}^1_A(M, M).$$

Sketch of the proof. Let $k[\epsilon] = k \oplus k\epsilon$ be the algebra of dual numbers, $\epsilon^2 = 0$. By definition, a tangent vector to