8. SEMI-SIMPLICIAL WEIL ALGEBRAS

8.1 OUTLINE. If one wants to apply the methods so far developed to complex analytic manifolds and algebraic varieties, one obstacle is the fact that no global connections need exist on holomorphic bundles or algebraic bundles \( P \). This is a difficulty occurring already if one wants to use the ordinary Chern-Weil construction for \( P \). But what one can use instead is a family of local connections on \( P \) restricted to the sets of a cohomologically trivial open covering of \( M \). Even in the smooth case connections are often given in this way, and a direct construction of the characteristic classes of \( P \) via these data seems desirable, regardless of the fact that they can be constructed by using a global connection in \( P \). If one wants to work with these data directly, one is then lead automatically to semi-simplicial methods, as the resulting invariants are defined via Čech cohomology. The basic idea for the construction of the generalized characteristic homomorphism \( \Delta_*(P) \) in this situation is then as follows ([KT 6,7,8,12]). Let \( \mathcal{U} = (U_j) \) be a \( \Gamma \)-acyclic open covering of the base space of the foliated \( G \)-bundle \( P \rightarrow M \). Then there exists a family \( \omega = (\omega_j) \) of local connections \( \omega_j \) on \( P|U_j \), adapted to the flat partial connection. \( \omega \) defines then a formal connection in the \( g \)-DG-algebra of Čech cochains \( \tilde{C}'(\mathcal{U}, \pi_*\Omega_P^\cdot) \) of \( \mathcal{U} \) with coefficients in the direct image of the De Rham complex \( \Omega_P^\cdot \) of \( P \). A semi-simplicial model \( W_1(g) \) of the Weil algebra \( W(g) \) is constructed, on which \( \omega \) defines a \( g \)-DG-algebra homomorphism

\[
{k_1(\omega) : W_1(g) \longrightarrow \tilde{C}'(\mathcal{U}, \pi_*\Omega_P^\cdot).}
\]

This generalized Weil homomorphism restricts then on \( H \)-basic elements for a subgroup \( H \) to the map giving rise to the characteristic
homomorphism $\Delta_*$ of $P$. A more detailed description of this 
construction is as follows.

First one defines a sequence of semi-simplicial models 
$W_s(g)$ for the Weil algebra $W(g)$. The construction of $W_1(g)$ is 
similar to the Amitsur complex of $W(g)$. $W_s(g)$ for $s > 1$ is 
obtained by an iterative process and $W_0(g) = W(g)$. The essential 
feature of these algebras is that they behave cohomologically exactly 
like the Weil algebra. More precisely, there are even filtrations 
$F_s(g)$ on $W_s(g)$ for $s \geq 0$ and canonical filtration preserving 
homomorphisms $\rho_s : W_s(g) \rightarrow W_{s-1}(g)$ for $s > 0$ inducing 
cohomology-isomorphisms of the associated graded algebras and hence 
also of the truncated algebras $W_s(g) = W_s/F_s^{2(k+1)}$, $s \geq 0$, 
$0 \leq k \leq \infty$. The same is true for $h$-basic subalgebras provided 
h $\subset g$ satisfies certain conditions (theorem 8.12). The algebra 
$W_1(g)$ is non-commutative except in trivial cases. Using this it 
is possible to show that the projection $\rho_1 : W_1(g) \rightarrow W(g)$ does 
not admit a splitting by a $g$-DG-algebra-homomorphism. However a 
splitting $\lambda : W(g) \rightarrow W_1(g)$ of $\rho_1$ by a $g$-DG-module map $\lambda$ 
does exist. This is the map of theorem 5.35.

Then one constructs the homomorphism $k_1(\omega)$ mentioned 
above. It is filtration preserving, where the filtration on the 
target complex $\tilde{C}(\mathcal{U}, \tau_* \Omega^*_P)$ is given by the ideals generated by 
the powers of $Q^*$ in $\tau_* \Omega^*_P$. For an $H$-structure of $P$ defined by 
a cross-section $s : M \rightarrow P/H$ the generalized characteristic homomorphism $\Delta_*$ is then also defined in this more general context. 
The same properties as in theorem 4.43 hold.

We finally describe explicitly some invariants associated 
to a foliated holomorphic vectorbundle whose first integral Chern 
class is zero.