Let $X = (\mathbb{R}^n, \| \cdot \|)$ be a normed space and $K = K(X) = \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$ its unit ball. We denote $|K| = \text{Vol} K$, where Vol means the standard Lebesgue volume on $\mathbb{R}^n$ equipped with the standard Euclidean structure. Let $\lambda_i \in \mathbb{R}$ and $x = (\lambda_i)_{i=1}^m$. Consider the following symmetric norm $\| \cdot \|_K$ on $\mathbb{R}^m$,

$$\|\|x\|\|_K = \int_{x \in K} \cdots \int_{x \in K} \| \sum_{i=1}^m \lambda_i x_i \| \frac{dx_1 \cdots dx_m}{|K|^m}.$$  

At the Denmark Conference on Probability in Banach spaces (June, 1986), the following question was asked (by V.M.):

Is it true that, for any $X$ (or, in other terms, for any centrally symmetric compact body $K \subset \mathbb{R}^n$ and the norm generated by this body), the symmetric norm $\| \cdot \|_K$ is, up to $\log n$, close to the standard Euclidean norm?

Moreover, is it true that under cotype condition on $X$ this norm is equivalent to the Euclidean norm?

Precisely:

Problem. Does there exist a constant $C = C(q, C_q(X))$ depending only on $2 \leq q < \infty$ and on the cotype $q$ constant $C_q(X)$, such that

$$\frac{1}{C} \left( \sum_{i=1}^m \lambda_i^2 \right)^{1/2} \leq \|\|x\|\|_K \leq C \left( \sum_{i=1}^m \lambda_i^2 \right)^{1/2}.$$  

Note that the question appears first as some kind of a generalization of the Khinchine inequality $\text{Ave}_{\epsilon_i = \pm 1} \left| \sum_{i=1}^m \epsilon_i \lambda_i \right| \approx \left( \sum_{i=1}^m \lambda_i^2 \right)^{1/2}$, complementary to the Kahane-type generalization: in the Kahane inequality the numbers $\lambda_i$ are replaced by vectors $x_i \in X$ and, as in the original Khinchine inequality, an equivalence between different $L_p$-norms is established:

$$\left\| \sum_{i=1}^m \epsilon_i x_i \right\|_{L_p(X)} \overset{\text{def}}{=} \left( \text{Ave}_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^m \epsilon_i x_i \right\|^p \right)^{1/p} \leq C(p) \left\| \sum_{i=1}^m \epsilon_i x_i \right\|_{L_1(X)}.$$
(Kahane [Ka] originally proved $C(p) \leq C \cdot p$ and later Kwapien [Kw] improved this to be $C(p) \leq C \sqrt{p}$; see [Ta] for a modern approach.)

In our generalization, we still let $\lambda_i$ be numbers but we replace the independent random variables $\varepsilon_i$ by vector valued random variables $x_i \in (K, \mu_{vol})$, where the probability measure $\mu_{vol}(A) = \frac{\text{Vol}A}{\text{vol}(K)}$ for any Borel subset $A \subset K$.

Note also that using a Borell lemma [Bo] in a standard way – see [MSch], Appendix III – we easily establish an equivalence for the different $L_p$-norms when $1 \leq p < \infty$:

$$
\left( \int_{\|x\|\leq K} \frac{\sum_{i=1}^{m} \lambda_i x_i}{|K|^m} d\|x\| \right)^{1/p} \leq C \cdot p \cdot \int_{\|x\|\leq K} \frac{\sum_{i=1}^{m} \lambda_i x_i}{|K|^m} d\|x\| = C \cdot p
$$

(we abbreviate $(x_i \in K)_{i=1}^{m} = \bar{x}$ for $\bar{x} \in \cap_{i=1}^{m} K$; likewise $dx_1 \cdot \cdots \cdot dx_m = d\bar{x}$).

It is clear at the present time, that the positive solution of the problem will imply numerous consequences in Local Theory.

In this note, we show a low bound in the most interesting case of $m \geq n$ and $\lambda_i = 1$.

Observe that in the special case of $X$ with an unconditional basis, the problem was solved by G. Pisier (unpublished). This case looks easy. Unfortunately, all consequences of the positive solution of the problem were well known in this case.

In what follows, we also consider a more general expression. Our norm $\| \cdot \|$ is defined by a fixed body $K$ (we emphasize this sometimes writing $\| \cdot \|_K$) but we integrate over different bodies (central symmetric) $(T_i \subset R^n)_{i=1}^{m}$:

$$
J(K; T_1, \ldots, T_m) = \int_{x_1 \in T_1} \cdots \int_{x_m \in T_m} \| \sum_{i=1}^{m} x_i \|_K \frac{dx_1 \cdots dx_m}{|T_1| \cdots |T_m|}.
$$

In the case $T_i = T$ for all $i = 1, \ldots, m$, we write simply $J(K; T, m)$.

**Theorem 1.1.** There exists a universal constant $c > 0$ such that for $m \geq n$

$$
J(K; T, m) \geq c \sqrt{m} |T|^{1/n} / |K|^{1/n}.
$$

In a general case (for $m = n$)

$$
J(K; T_1, \ldots, T_n) \geq c \sqrt{n} (\cap_{i=1}^{n} |T_i|)^{1/n} / |K|^{1/n}.
$$

**Corollary 1.2.**

$$
\| \|x\|\|_{K; T} = \int_{\|x\|\leq T} \| \sum_{i=1}^{n} \lambda_i x_i \|_K \frac{d\|x\|}{|T|} \geq c \sqrt{n} (\cap |\lambda_i|)^{1/n} \left( \frac{|T|}{|K|} \right)^{1/n}.
$$