4. Divisorial cycles on a normal projective variety $V/k \ (\dim(V) = r \geq 1)$

**Def. 4.1**: A prime divisorial cycle on $V/k$ is a subvariety $W$ of $V$ such that $W$ is defined and irreducible over $k$ and $\dim(W) = r - 1$.

**Def. 4.2**: If $K$ is a finitely generated extension field of $k$ and $t.d.(K/k) = r$, then a prime divisor of $K/k$ is a valuation $v$ of $K/k$ such that the residue field of $v$ has transcendence degree $r - 1$ over $k$.

**Prop. 4.3**: If $\Gamma$ is a prime divisorial cycle on $V$, then $\mathcal{O}_\Gamma(V/k)$ is the valuation ring of a prime divisor $\Gamma$ of $k(V)/k$.

Furthermore, $\Gamma$ is a discrete, rank 1 valuation, and the residue field of $\Gamma$ is $k(\Gamma)$.

Proof: Let $H$ be defined by $Y_o = 0$. Since we can assume $\Gamma \notin H$, let $V_g = \overline{\Gamma}(\cap \Gamma \cap H)$ and $\Gamma_a = \Gamma - (\Gamma \cap H)$. Then $\dim(\Gamma_a) = r$ and $\dim(\Gamma_a) = r - 1$. Let $R = k[V_a]$; then $R$ is integrally closed and noetherian. If $\mathfrak{p}$ is the prime ideal of $\Gamma_a$ in $R$, then $\mathfrak{p}$ is a minimal prime ideal in $R$ since $\dim(\Gamma_a) = r - 1$. We have $\mathcal{O}_\Gamma(V/k) = \mathcal{O}_\Gamma(V_a)/k = R_\mathfrak{p}$, where $R_\mathfrak{p}$ is integrally closed and noetherian because $R$ is. Furthermore, since $\mathfrak{p}$ is minimal in $R$, $R_\mathfrak{p}$ is the only proper prime ideal in $R_\mathfrak{p}$. We have thus shown that $R_\mathfrak{p}$ is a Dedekind domain with only one proper prime ideal. Hence $R_\mathfrak{p}$ is a discrete valuation ring of rank 1.

Since $R/\mathfrak{p} = k[\Gamma_a]$ and $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the quotient field of $R/\mathfrak{p}$, we have the last statement.
The following is an easy consequence of Prop. 4.3 (but we shall not prove it here):

Cor. 4.4: With the assumptions and notations of Def. 4.2, every prime divisor of $K/k$ is a discrete, rank 1 valuation, and its residue field is also finitely generated over $k$.

Def. 4.5: If $k(V)$, $\neq 0$, the integer $v_\Gamma(\xi)$ is the order of $\xi$ on $\Gamma$. $\Gamma$ is a prime null cycle of $\xi$ if $v_\Gamma(\xi) > 0$, a prime polar cycle if $v_\Gamma(\xi) < 0$. If $\xi = 0$, we define $v_\Gamma(0) = +\infty$.

Prop. 4.6: Let $Q \in V \subset \mathbb{S}^n$, and let $\xi \in k(V)$. If no prime polar cycle of $\xi$ passes through $Q$, then $\xi$ is regular at $Q$.

Proof: Let $\sigma = \sigma(V/k)$. A prime cycle $\Gamma$ corresponds to a minimal prime ideal $\mathfrak{p}$ of $\sigma$. By assumption $\xi \notin \mathfrak{p}$, hence $\xi \notin \cap \sigma_\mathfrak{p}$ where $\mathfrak{p}$ runs through the minimal prime ideals of $\sigma$. Since $V$ is normal, $\sigma$ is integrally closed, and so $\cap \sigma_\mathfrak{p} = \sigma$. Hence $\xi \in \sigma$.

Prop. 4.7: If $\xi$ has no polar cycles, then $\xi$ is a constant, i.e. $\xi$ is algebraic over $k$.

Proof 1: If $\xi$ has no polar cycles, then $\xi \in \bigcap_{Q \in V} \sigma(V/k)$; and if $v$ is any valuation of $k(V)/k$, then $v(\xi) > 0$. Assume $\xi$ is transcendental over $k$, and consider $\eta$ where $\eta = 1/\xi$. By the extension theorem there exists a valuation $v$ such that $R_v \supset k[\eta]$ and $M_v$ contains the principal ideal $(\eta)$. Since $v(\eta) > 0$, $v(\xi) < 0$ which is a contradiction. Hence $\xi$ is algebraic over $k$.

Proof 2: We have $\bigcap_{Q \in V_a} \sigma(V_a/k) = k[V_a]$. Let $H_i$ be the hyperplane given by $Y_i = 0$, and let $V(1) = V - (V \cap H_i)$ where $i = 0, \ldots, n$. $k[V] = k[y_0, \ldots, y_n]$ and $k[V_a(\sigma)] = k[x_1, \ldots, x_n]$ where $x_i = y_i/y_0$. 