In this section, we consider approximation by weighted polynomials of the form $P_n(x)W(a_n x)$, or more precisely of the form $P_n(x)/H(a_n x)$, where $H(x)$ is a certain entire function that behaves like $W^{-1}$ on the real line. Weighted polynomial approximations of the form $P_n(x)W(a_n x)$ were first considered by Knopfmacher, Lubinsky and Nevai [21] and Lubinsky and Saff [32]. In the special case $W(x) = \exp(-|x|^\alpha)$, approximation by $P_n(x)W(a_n x)$ is equivalent to approximation by $P_n(x)\tilde{W}_n(x)$. The latter type of weighted polynomial had previously been considered in the framework of incomplete polynomials, and Saff [63] and Mhaskar and Saff [49] formulated certain conjectures in this connection. In the case $W(x) = \exp(-|x|^\alpha)$, the results of [32] largely resolved Saff's conjecture. For Erdős type weights, the problem has been considered by Knopfmacher and Lubinsky [20], and Lubinsky [29].

An important tool in [32] was the entire function

$$G_Q(x) := 1 + \sum_{n=1}^{\infty} (x/q_n)^{2n} n^{-1/2} e^{2Q(q_n)}.$$  

where $q_n$ is the root of $n = q_n Q'(q_n)$, for $n$ large enough. This function was introduced, and its behaviour as $|x| \to \infty$ was investigated in [27, 28]. Rather than $G_Q$ itself, we shall use $G_{Q/2}$. Since working with $Q/2$ rather than $Q$ involves replacement of $q_n$ by $q_{2n}$, we see that

$$G_{Q/2}(x) = 1 + \sum_{n=1}^{\infty} (x/q_{2n})^{2n} n^{-1/2} e^{Q(q_{2n})}.$$  

One of the two main results of this section is:

**Theorem 11.1**

Let $W(x) = e^{-Q(x)}$, where $Q(x)$ is even and continuous in $\mathbb{R}$, $Q''(x)$ exists and is continuous in $(0, \infty)$, $Q'(x)$ is positive in $(0, \infty)$, while for some $C_1, C_2 > 0$,
Let $H(x)$ be an even entire function with non-negative Maclaurin series coefficients satisfying

$$C_3 \leq H(x) \leq C_4, \quad x \in \mathbb{R},$$

for some $C_3, C_4 > 0$. For example, we may choose $H = G_{Q/2}$. Let $g(x)$ be a function continuous in $\mathbb{R}$, with $g(x) = 0$, $|x| \geq 1$, and let $\{k_n\}_{n=1}^{\infty}$ be a sequence of non-negative integers satisfying

$$\lim_{n \to \infty} k_n/n = 0.$$

Then there exist $P_n \in P_{n-k_n}$, $n = 1, 2, 3, \ldots$, such that

$$\lim_{n \to \infty} \|g(x) - P_n(x)/H(a_n x)\|_{L^\infty(\mathbb{R})} = 0.$$

The above result was proved as Theorem 4.1 in [32] when (11.3) is replaced by [32, eqn.(2.13)]

$$\lim_{|x| \to \infty} (xQ'(x))'/Q'(x) = \alpha > 0.$$ 

This is asymptotically a stronger condition than (11.3), forcing

$$\lim_{|x| \to \infty} \log Q(x)/\log |x| = \alpha,$$

but does not place restrictions on $Q(x)$ for small $|x|$. When we assume more about $g(x)$, we can obtain rates of convergence: Recall that $G(\epsilon; \theta)$ is the region defined by (9.2) and illustrated in Figure 9.1.

**Theorem 11.2**

Let $W$ and $H$ be as in Theorem 11.1. Let $h(x)$ be bounded and analytic in $(-1,1) \cup G(\epsilon_0; (\pi/3)+\delta)$ for some $\epsilon_0 > 0$ and $0 < \delta < \pi/6$ and let

$$\|h\| := \sup \{|h(t)| : t \in G(\epsilon_0; (\pi/3)+\delta)\}.$$ 

Let $\eta \geq 0$ and

$$g(x) := \begin{cases} h(x) (1 - x^2)^{\eta}, & x \in (-1,1), \\ 0, & x \in \mathbb{R}\setminus(-1,1). \end{cases}$$

Let