12. Approximation by Certain Weighted Polynomials, II.

Whereas in the previous section we considered approximation by weighted polynomials of the form $P_n(x)/H(a_n x)$, in this section we consider approximation by $P_n(x)W(a_n x)$, or more generally $P_n(x)W(c_n x)$. This requires replacing the reciprocal of the entire function $H(x)$ by the weight itself. In the case when $W(x) = \exp(-x^m)$, $m$ a positive even integer, one can choose $H = 1/W$. But in the general case, we have to choose $H = G_q/2$. Furthermore, a considerable amount of effort is involved in the transition from $1/H$ to $W$. First, an analogue of Theorem 11.1:

Theorem 11.1

Let $W(x) := e^{-Q(x)}$, where $Q(x)$ is even and continuous in $\mathbb{R}$, $Q''(x)$ exists and is continuous in $(0, \infty)$, and $Q'(x)$ is positive in $(0, \infty)$, while for some $C_1, C_2 > 0$,

$$
(12.1) \quad C_1 \leq (xQ'(x))'/Q'(x) \leq C_2, \quad x \in (0, \infty).
$$

Suppose, further, that $Q'''(x)$ exists and is continuous for $x$ large enough, with

$$
(12.2) \quad x^2|Q'''(x)|/Q'(x) \leq C_3, \quad x \in (c_4, \infty).
$$

Let $a_n = a_n(W)$ for $n = 1, 2, 3, \ldots$, and let

$$
(12.3) \quad c_n := a_n(1 + \epsilon_n), \quad n = 1, 2, 3, \ldots,
$$

where $\{\epsilon_n\}_1^\infty$ is a sequence of real numbers satisfying

$$
(12.4) \quad \limsup_{n \to \infty} \epsilon_n n^{1/2} \leq 0
$$

and

$$
(12.5) \quad \liminf_{n \to \infty} \epsilon_n (\log n)^2 \geq 0.
$$

Let $\{k_n\}_1^\infty, k_n \leq n$, be a sequence of non-negative integers satisfying
(12.6) \( \lim_{n \to \infty} k_n n^{-1/2} = 0. \)

Let \( g(x) \) be positive and continuous in \([-1, 1]\). Then there exists \( P_n \in P_{n-k_n}, \) \( n=1, 2, 3, \ldots \), satisfying

(12.7) \( \| P_n(x)W(c_n x) \|_{L^\infty([-1,1])} \leq C_5, \quad n=1, 2, 3, \ldots, \)

such that

(12.8) \( \lim_{n \to \infty} P_n(x)W(c_n x) = g(x), \)

uniformly in compact subsets of \( \{ x : 0 < |x| < 1 \} \), and

(12.9) \( \lim_{n \to \infty} P_n(x)W(c_n x) = 0, \)

uniformly in closed subsets of \( \{ x : |x| > 1 \} \). Furthermore,

(12.10) \( \lim_{n \to \infty} \int_{-1}^{1} \left| \log |P_n(x)W(c_n x)/g(x)| \right| (1 - x^2)^{-1/2} \, dx = 0. \)

Finally, if \( \{ k_n \}^\infty_1 \) and \( \{ \epsilon_n \}^\infty_1 \) are further restricted so that

(12.11) \( \lim_{n \to \infty} \epsilon_n (n/(\log n))^{2/3} = -\infty, \)

and

(12.12) \( \limsup_{n \to \infty} k_n n^{-1/3} (\log n)^{-2/3} < \infty. \)

we may also ensure that

(12.13) \( |P_n(x)W(c_n x)| \geq C_6, \quad x \in [-1, 1], \; n=1, 2, 3, \ldots. \)

We remark that the polynomial above has complex coefficients. The reason for this is that we add a small imaginary part to ensure that \( \log |P_n(x)W(c_n x)| \) is not too small near \( x = \pm 1 \). One may also think of \( |P_n(x)| \) as \( S_{2n}(x)^{1/2} \), where \( S_{2n}(x) \) is a polynomial non-negative in \( \mathbb{R} \), of degree at most \( 2(n - k_n) \). The following result establishes a certain precise interval in which we may approximate positive continuous functions by weighted polynomials \( P_n(x)W(a_n x) \).