12. Approximation by Certain Weighted Polynomials, II.

Whereas in the previous section we considered approximation by weighted polynomials of the form \( P_n(x)/H(a_n x) \), in this section we consider approximation by \( P_n(x)W(a_n x) \), or more generally \( P_n(x)W(c_n x) \). This requires replacing the reciprocal of the entire function \( H(x) \) by the weight itself. In the case when \( W(x) = \exp(-x^m) \), \( m \) a positive even integer, one can choose \( H = 1/W \), but in the general case, we have to choose \( H = CQ/2 \). Furthermore, a considerable amount of effort is involved in the transition from \( 1/H \) to \( W \). First, an analogue of Theorem 11.1:

**Theorem 12.1**

Let \( W(x) := \exp(-Q(x)) \), where \( Q(x) \) is even and continuous in \( \mathbb{R} \), \( Q''(x) \) exists and is continuous in \( (0, \infty) \), and \( Q'(x) \) is positive in \( (0, \infty) \), while for some \( C_1, C_2 > 0 \),

\[
C_1 \leq (xQ'(x))/Q'(x) \leq C_2, \quad x \in (0, \infty).
\]

Suppose, further, that \( Q'''(x) \) exists and is continuous for \( x \) large enough, with

\[
x^2|Q'''(x)|/Q'(x) \leq C_3, \quad x \in (C_4, \infty).
\]

Let \( a_n = a_n(W) \) for \( n = 1, 2, 3, \ldots \), and let

\[
c_n := a_n(1 + \epsilon_n), \quad n = 1, 2, 3, \ldots
\]

where \( \{\epsilon_n\}_1^\infty \) is a sequence of real numbers satisfying

\[
\limsup_{n \to \infty} \epsilon_n n^{1/2} \leq 0
\]

and

\[
\liminf_{n \to \infty} \epsilon_n (\log n)^2 \geq 0.
\]

Let \( \{k_n\}_1^\infty \), \( k_n \leq n \), be a sequence of non-negative integers satisfying...
(12.6) \[ \lim_{n \to \infty} k_n n^{-1/2} = 0. \]

Let \( g(x) \) be positive and continuous in \([-1,1]\). Then there exists \( P_n \in P_{n-k_n} \), \( n=1,2,3,... \), satisfying

(12.7) \[ \| P_n(x)W(c_n x) \|_{L^\infty[-1,1]} \leq C_5, \quad n=1,2,3,..., \]

such that

(12.8) \[ \lim_{n \to \infty} P_n(x)W(c_n x) = g(x), \]

uniformly in compact subsets of \( \{ x: 0 < |x| < 1 \} \), and

(12.9) \[ \lim_{n \to \infty} P_n(x)W(c_n x) = 0, \]

uniformly in closed subsets of \( \{ x: |x| > 1 \} \). Furthermore,

(12.10) \[ \lim_{n \to \infty} \int_{-1}^{1} |\log |P_n(x)W(c_n x)/g(x)|| (1 - x^2)^{-1/2} dx = 0. \]

Finally, if \( \{ k_n \}_1^\infty \) and \( \{ \epsilon_n \}_1^\infty \) are further restricted so that

(12.11) \[ \lim_{n \to \infty} \epsilon_n \left( n/(\log n) \right)^{2/3} = -\infty, \]

and

(12.12) \[ \limsup_{n \to \infty} k_n n^{-1/3} (\log n)^{-2/3} < \infty, \]

we may also ensure that

(12.13) \[ |P_n(x)W(c_n x)| \geq C_6, \quad x \in [-1,1], \quad n=1,2,3,... \]

We remark that the polynomial above has complex coefficients.

The reason for this is that we add a small imaginary part to ensure that \( \log |P_n(x)W(c_n x)| \) is not too small near \( x = \pm 1 \). One may also think of \( |P_n(x)| \) as \( S_{2n}(x)^{1/2} \), where \( S_{2n}(x) \) is a polynomial non-negative in \( \mathbb{R} \), of degree at most \( 2(n-k_n) \). The following result establishes a certain precise interval in which we may approximate positive continuous functions by weighted polynomials \( P_n(x)W(a_n x) \).