Removing Index-Zero Singularities with $C^1$-small Perturbations.

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1. Introduction.

Let $D^n$ be an $n$-dimensional disk with boundary an $(n-1)$ sphere, $S^{n-1}$. It is well known that a continuous $f : S^{n-1} \rightarrow S^{n-1}$ is extendable to a continuous $F : D^n \rightarrow S^{n-1}$ if and only if the degree of $f$ is zero. Furthermore, given a continuous mapping $F : D^n \rightarrow \mathbb{R}^n$, a point $p$ in the interior of $D^n$ and a closed disk $U^n$ such that $F^{-1}(F(p)) \cap U^n = \{p\}$ and the local degree of $F|U^n$ at $F(p)$ is zero, then for every $\varepsilon > 0$ there exists a mapping $G : D^n \rightarrow \mathbb{R}^n$ such that

(i) $G$ is $\varepsilon$-close to $F$ in the $C^0$ topology,
(ii) $G$ is equal to $F$ on $D^n \setminus U^n$,
(iii) $F(p)$ is not in the image of $G|U^n$.

Now, suppose $M^n$ is a $C^k$ manifold of dimension $n$, $k$ and $n$ at least one. Let $X$ be a tangent vector field on $M$ and $U$ a closed $n$-disk in $M$ about $p$ such that $X|U$ is zero only at $p$ and the index of $p$ as a stationary point of $X$ is zero. Since one can consider $X|U$ as a mapping $U \rightarrow \mathbb{R}^n$, the above statement implies that for every $\varepsilon > 0$, there exists a vector field $Y$ on $M$ so that

(i) $Y$ is $\varepsilon$-close to $X$ in the $C^0$ topology,
(ii) $Y$ is equal to $X$ outside $U$,
(iii) $Y$ has no zeros in $U$.

In other words, index zero singularities can be removed with $C^0$-small perturbations.

Whether or not this phenomenon persists for $C^{k+2}$ vector fields $X$ with perturbations to vectorfields $Y$ nearby in the $C^k$ topology is an important question in the study of stability and bifurcations of dynamical systems. See, for example, Question 4.1 in Hirsch [1]. In this paper, we conjecture that this phenomenon indeed persists for all $k$ and $n$ and settle the conjecture for $k = 1$ and $n = 2$. 
Theorem 1. Let $M$ be a smooth 2-manifold with $X$ a $C^k$ vector field on $M$, $k \geq 1$. Let $U$ be a closed 2-disk in $M$ with $X|U$ zero only at $p \in U$ and the index of $X$ at $p$ equal to zero. Then, for every $\varepsilon > 0$, there exists a $C^k$ vector field $Y$ such that

(i) $Y$ is $\varepsilon$-close to $X$ in the $C^1$-topology,
(ii) $Y$ equals $X$ outside $U$,
(iii) $Y$ has no zeroes in $U$.

The next theorem states the corresponding result for fixed points of maps or diffeomorphisms. Let $k \geq 1$ and let $C^{k,1}(M^2)$ be the space of $C^k$ mappings of the manifold $M^2$ to itself with the fine $C^1$ topology; see Munkres [2].

Theorem 2. Let $h \in C^{k,1}(M^2)$ with $p$ an isolated fixed point of $h$ of index zero. Let $N$ be a neighbourhood of $h$ in $C^{k,1}(M^2)$ and $U$ a neighbourhood of $p$ containing no other fixed point of $h$. Then, there is an $\hat{h} \in N$ with $\hat{h} = h$ outside $U$ and with $\hat{h}$ fixed point free in $U$. Furthermore, if $h$ is a diffeomorphism, $\hat{h}$ can be chosen to be a diffeomorphism.

Theorem 3 describes the same phenomenon from another point of view.

Theorem 3. Let $P$ be a smooth 4-dimensional manifold, and let $L_1$ and $L_2$ be smooth submanifolds of $P$ of dimension $\kappa$ and $4-\kappa$ respectively. Suppose $x \in L_1 \cap L_2$ such that (a) there is a neighbourhood $U$ of $x$ in $P$ with $L_1 \cap L_2 \cap U = \{x\}$ and (b) the local intersection number of $L_1$ and $L_2$ at $x$ is zero. Then, arbitrarily $C^1$-close to $L_2$ there is a smooth $(4-\kappa)$-dimensional submanifold of $P$, say $\hat{L}_2$, such that (i) $L_2 = \hat{L}_2$ outside $U$ and (ii) $\hat{L}_2 \cap L_1 \cap U = \emptyset$.

Before proceeding with the proofs of these Theorems, we first make a few simplifications and normalizations. These Theorems are all local results and the proofs of Theorems 1 and 2 reduce immediately to the