SOME INTEGRAL CALCULUS BASED ON EULER CHARACTERISTIC

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Sometimes in appears to be useful to consider the Euler characteristic as a (finitely-additive) measure and, in particular, to integrate with respect to it. The following notes are gathered to justify this point of view.

GENERALITIES

1. Integration with respect to a finitely-additive measure.

Let X be a set, $\mathcal{A}$ a collection of subsets of X closed with respect to be operations of (finite) union and (finite) intersection. Let R be a commutative ring and $\mu : \mathcal{A} \longrightarrow R$ a function with the property

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

for $A, B \in \mathcal{A}$. Let $\mathcal{F}(X, \mathcal{A}, \mu)$ denote the ring of finite R-linear combinations of characteristic functions of elements of $\mathcal{A}$. It is not difficult to prove the following assertion.

1A. For any $B \in \mathcal{A}$ there is a well defined functional $\mathcal{F}(X, \mathcal{A}, \mu) \longrightarrow R : f \longmapsto \int_B f(x)\, d\mu(x)$ with $\int_B f(x)\, d\mu(x) = \sum \lambda_A \mu(A \cap B)$ if $f = \sum \lambda_A \mathbb{1}_A$.

2. Integration with respect to Euler characteristic.

Below the construction of Section 1 is applied in the following

more specialized situations: X is a topological space, each element $A \in \mathcal{A}$ has a well defined Euler characteristic $\chi(A)$. Below elements of $\mathcal{A}$ are called t a m e s e t s . We take $R = \mathbb{Z}$ and $\mu = \chi$ and abbreviate $\mathcal{F}(X, \mathcal{A}, \mathbb{Z})$ to $\mathcal{F}(X)$. Thus on $\mathcal{F}(X)$ we have a well defined integration-operation which assigns to a function $f = \sum_{A \in \mathcal{A}} \lambda_A 1\!\!1_A$ and a set $B \in \mathcal{A}$ the number

$$\int_B f(x)\, d\chi(x) = \sum \lambda_A \chi(A \cap B)$$

The basic example of such a situation: X a projective algebraic variety over $\mathbb{R}$ or $\mathbb{C}$, $\mathcal{A}$ an algebra of closed semi-algebraic sets.

3. Fubini theorem and Riemann-Hurwitz theorem.

Another fundamental property of the Euler characteristic is its multiplicativity: $\chi(X \times Y) = \chi(X)\chi(Y)$. It implies that if $E \to B$ is a locally-trivial fibration with fibre F then $\chi(E) = \chi(B)\chi(F)$

To use and extend this property let us introduce the following notion. Let X and Y be spaces with algebras $\mathcal{A}$ and $\mathcal{B}$ of tame sets. A map $\varphi : X \to Y$ is said to be t a m e (with respect to $\mathcal{A}$ and $\mathcal{B}$) if (i) $\varphi^{-1}(y) \in \mathcal{A}$ for any $y \in Y$ and (ii) for any $A \in \mathcal{A}$ there exist $S_0 \subset S_1 \subset \ldots$ with $S_i \in \mathcal{B}$ such that the maps

$$\varphi^{-1}(S_i \setminus S_{i-1}) \cap A \longrightarrow S_i \setminus S_{i-1}$$

defined by φ are locally-trivial fibrations.

It is well known that any regular map of a real algebraic variety to another one is tame with respect to the algebras of closed semi-algebraic sets.

The following theorem obviously follows from the multiplicativity property of χ stated above and extends it.

3.A. (Fubini-type theorem). If $\varphi : X \to Y$ is a tame map and $f \in \mathcal{F}(X)$ then