On Degenerate Monge-Ampere Equations in Convex Domains

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§1 Introduction

In [1] the author solved degenerate equations of Monge-Ampere type in \( \mathcal{C}^{1,1}(\Omega) \) with some restrictions on Dirichlet data or on degeneracy. N. Trudinger in [2] discussed degenerate Monge-Ampere equations in \( \mathcal{C}^{1,1}(\Omega) \cap \mathcal{C}^{0,1}(B) \) without these restrictions but only in balls \( B \).

Enlightened by [2] we extend the results in [2] to the equations of Monge-Ampere type in a general convex domain. The idea comes from Krylov's work [3] in which he considered elliptic operator on manifolds and applied it to getting the boundary estimates for derivatives of solutions of Dirichlet problem with homogeneous boundary values. The method given in this paper is much simpler than those in [3], and it is used to find interior second derivative estimates for general Dirichlet data.

Let \( \Omega \) be a convex domain in \( \mathbb{R}^n \). In \( \Omega \) we consider the Dirichlet problem

\[
\begin{align*}
\text{det} D^2 u &= f(x, u, Du) \quad \text{in } \Omega, \\
\quad u &= \varphi \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( Du \) and \( D^2 u \) are the gradient and the Hessian matrix of \( u \) respectively.

Set \( g(x, z, p) = \sqrt{|f(x, z, p)|} \). Assume \( f(x, z, p) \) satisfies

(F1) \( g \geq 0 \) in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \) and \( g \in \mathcal{C}^{1,1}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \).

(F2) \( g_z \geq 0 \) in \( \Omega \times \mathbb{R} \times \mathbb{R}^n \).

(F3) There exists a constant \( N \) and \( \tilde{g} \in \mathcal{C}^1(\Omega) \), \( h \in L^1_{\text{loc}}(\mathbb{R}^n) \) such that

\[
f(-, -N, p) \leq \tilde{g}(x)/h(p), \quad \forall x \in \Omega, p \in \mathbb{R}^n,
\]

where

\[
\int_{\Omega} \tilde{g} < \int_{\mathbb{R}^n} h, \quad h^{-1} \geq \mu_0 > 0, \quad \forall p \in \mathbb{R}^n.
\]

(F4) There exist non-negative constants \( \mu, \alpha \) and \( \beta \) such that

\[
f(x, \varphi(x), p) \leq \mu d^\beta (1 + |p|^2)^{\alpha/2}, \quad \forall x \in \mathcal{N}, p \in \mathbb{R}^n.
\]

where \( d = \text{dist}\{x, \partial \Omega\} \), \( \mathcal{N} \) is a neighborhood of \( \partial \Omega \) and \( \beta \geq \alpha - n - 1 \).

(F5) \( g(x, z, p) \) is convex with respect to \( p \).

In this paper we shall prove the following existence theorem:

**Theorem 1.1** Let \( \Omega \) be a convex domain in \( \mathbb{R}^n \) such that there exists a concave function \( \varphi \in \mathcal{C}^{3,1}(\Omega) \) satisfying

\[
\begin{align*}
\varphi &= 0, \quad |D\varphi| \neq 0 \quad \text{on } \partial \Omega, \\
D^2 \varphi &\leq -I, \quad \text{in } \Omega
\end{align*}
\]

where \( I \) is the unit matrix, and let \( f \) satisfy (F1)-(F5). Assume \( \varphi \in \mathcal{C}^{1,1}(\Omega) \). Then Dirichlet problem (1), (2) has the unique convex solution \( u \in \mathcal{C}^{1,1}(\Omega) \cap \mathcal{C}^{0,1}(\Omega) \).

§2 Elliptic operators on manifolds

Consider the elliptic operators

\[
Lu = a^{ij} D_{ij} u + b^i D_i u
\]
where summation convention is used and the matrix \((a^{ij})\) satisfies
\[
(a^{ij}) \geq 0.
\]

The following maximum principle of elliptic operators on manifolds is due to Krylov [3]. Here we state only a special case.

**Lemma 1.2** Let \(\Psi \in C^\infty_0(\Omega)\) and \(M = \{x \in \Omega|\Psi(x) = 0\}\). Assume \([D\Psi] \neq 0\) on \(M\) and \(L\Psi = L\Psi^2 = 0\) on \(M\). If \(u \in C^\infty_0(\Omega)\) takes its local maximum at \(x_0 \in M\) with respect to the topology of \(M\), then \(Lu(x_0) \leq 0\).

**Proof** Consider the neighborhood \(V\) of \(x_0\) in \(\Omega\). Make the transformation \(x \to \bar{x} = (\Psi(x), x_2, \cdots, x_n)\) which is non-degenerate in \(V\). The manifold \(M \cap V\) is mapped into the hyperplane \(\bar{x}_1 = 0\) and by the hypotheses of the lemma the operator \(L\) is changed to
\[
\hat{L}u = \sum_{i,j=2}^n a^{ij} \bar{D}_{ij}u + \sum_{i=2}^n b^i \bar{D}_i u
\]
where \(\bar{D}_i = \frac{\partial}{\partial \bar{x}_i}\). The conclusion is obvious now.

**Remark** The conditions \(L\Psi = L\Psi^2 = 0\) are equivalent to \(L\Psi = a^{ij} D_i \Psi D_j \Psi = 0\).

### §3 The proof of Theorem 1.1

In [1] we showed the proof of Theorem 1.1 can be referred to the estimation of \(D^2 u\). Without loss of generality we can suppose \(u\) has enough smoothness and it is strictly convex. Write the equation (1.1) in the form
\[
(3.1) \quad \Phi(D^2 u) = [\det D^2 u]^{1/n} = \varrho(x,u,Du)
\]
By [1] we know
\[
(3.2) \quad \Phi^{ij}(r) = \frac{\partial \Phi}{\partial r_{ij}}(r) = \frac{1}{n} \Phi^{-1}
\]
\[
(3.3) \quad \frac{\partial^2 \Phi}{\partial r_{ij} \partial r_{kl}} = \frac{1}{n} \Phi[-r^{ik}r^{jl} + n^{-1}r^{ij}r^{kl}]\]
where \((r^{ij}) = r^{-1}\) and
\[
(3.4) \quad \det(\Phi^{ij}) = n^{-n}, \quad Tr(\Phi^{ij}) \geq 1.
\]
Consider \(2n + 1\)-dimensional space \((x, \xi, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\). Set
\[
(3.5) \quad \Psi(x, \xi, t) = \psi(\xi) - t\psi^{1/2},
\]
where \(\psi\) is the function defined in Theorem 1.1. Hereafter we denote
\[
\psi_i = D_i \psi, \quad \psi(\xi) = \psi_i \xi_i, \quad \psi(\xi)(\xi) = \psi_{ij} \xi_i \xi_j.
\]
Define the manifold
\[
(3.6) \quad M = \{(x, \xi, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}|\Psi(x, \xi, t) = 0\}.
\]
We can observe that for \((x, \xi, t) \in M\) and \(x \in \partial \Omega\) the direction \(\xi\) is tangential to \(\partial \Omega\) at \(x\).