In this paper, we briefly survey selected recent developments and present some new results in the area of $L^2$ harmonic forms on complete Riemannian manifolds. In light of studies of $L^2$ cohomology relating to singular varieties, discrete series representations of Lie groups, arithmetic quotients of symmetric spaces among others, our discussion will be rather limited, focussing only on aspects of $L^2$ harmonic forms in global Riemannian geometry. This paper is partly intended as a completion of the announcement [1]. One may refer to [13] for a previous survey in this area.

Throughout the paper, all manifolds will be connected, complete, oriented Riemannian manifolds, of dimension $n$.

§1. $L^2$ cohomology and $L^2$ harmonic forms.

[1.1] Let $\Delta_M$ denote the Laplace-Beltrami operator acting on $C^\infty$ $p$-forms $C^\infty(\Lambda^p(M))$ on the manifold $M$. The space of $L^2$ harmonic $p$-forms $\mathcal{H}^p_2(M)$ consists of those forms $\omega \in C^\infty(\Lambda^p(M))$ such that $\Delta_M \omega = 0$ and $\omega \in L^2$, i.e. $||\omega||^2 = \int_M \omega \star \omega < \infty$, where $\star : \Lambda^p(M) \to \Lambda^{n-p}(M)$ is the Hodge $\star$ operator. The regularity theory for elliptic operators implies that $\mathcal{H}^p_2(M)$ is a Hilbert space with $L^2$ inner product.

If $M$ is compact, the Hodge theorem implies that $\mathcal{H}^p_2(M) \cong H^p(M,\mathbb{R})$ so that these spaces are topological invariants of $M$. One guiding problem is to understand to what extent this remains true for non-compact manifolds. Note that since $\Delta$ and $\star$ commute, $\star$ induces on isomorphism $\mathcal{H}^p_2(M) \cong \mathcal{H}^{n-p}_2(M)$ (representing Poincaré duality for $M$ compact). In particular, $\star$ induces an automorphism of $\mathcal{H}^n_2(M)$ with $\star^2 = (-1)^{n/2}$. Since $\star$ is a conformal invariant on $n/2$-forms, one obtains the important fact that the Hilbert space structure on $\mathcal{H}^n_2(M)$ depends only on the conformal structure of $M$.

A well-known result of Andreotti-Vesentini [10] implies that

$$\mathcal{H}^p_2(M) = \{\omega \in C^\infty(\Lambda^p(M)) \cap L^2 : d\omega = 0 \text{ and } \delta \omega = 0\},$$

where $d$ is the exterior derivative and $\delta$ its formal adjoint. Thus one has a natural map

$$\mathcal{H}^p_2(M) \to H^p_{deR}(M).$$

We now relate the space of $L^2$ harmonic forms on $M$ to its $L^2$ cohomology. The simplest definition of $L^2$ cohomology is

*Partially supported by NSF Grant DMS-8701137
where \( \ker dp = \{ \omega \in C^0(\Lambda^p M) \cap L^2 : \text{d}\omega = 0 \} \), \( \text{Im } dp^{-1} = \{ \eta \in C^0(\Lambda^{p-1} M) \cap L^2 : \text{d}\eta = \eta \} \),

for some \( \alpha \in \text{dom } dp^{-1} \) with \( \text{dom } dp^{-1} = \{ \alpha \in C^0(\Lambda^{p-1} M) \cap L^2 : \text{d}\alpha \in L^2 \} \). Clearly there is a natural map

\[
\iota_# : H^p(M) \to H^p(M)
\]

and one says the Strong Hodge Theorem holds if \( \iota_# \) is an isomorphism. Cheeger [6] has shown that \( \iota_# \) is always an injection (since \( M \) is assumed complete). However, in many cases \( \iota_# \) is not surjective. For example, it is easily calculated that \( H^0(\mathbb{R}) = \{ \mathbb{R} \} \), but \( H^1(\mathbb{R}) \) is infinite dimensional (c.f. [6]).

Define the reduced \( L^2 \) cohomology by

\[
\overline{H}^p(M) = \frac{\ker d}{\text{Im } d^{-1}},
\]

where the closure is taken in \( L^2 \) and \( d \) is the strong closure of \( d \) in \( L^2 \), i.e., \( d\alpha = \beta \) if \( \exists \alpha_i \in \text{dom } d \) such that \( \alpha_i \to \alpha \) and \( d\alpha_i \to \beta \) in \( L^2 \). There is a natural surjection \( H^p(M) \to \overline{H}^p(M) \) and we have the basic fact [6] that

\[
H^p(M) \cong \overline{H}^p(M)
\]

for any complete Riemannian manifold \( M \). (1.3) may be viewed as a non-compact Hodge theorem: the reduced \( L^2 \) de Rham cohomology of \( M \) is isomorphic to the space of \( L^2 \) harmonic forms. For a de Rham-type theorem relating \( \overline{H}^p(M) \) to the simplicial \( L^2 \) cohomology of \( M \), c.f. [12].

An immediate consequence of (1.3) is that \( \mathcal{M}^p(M) \) (up to equivalence) depends only on the quasi-isometry class of the metric on \( M \), since it is easily verified that the topology on \( \overline{H}^p(M) \) is a quasi-isometry invariant. In particular, if \( M \) is a (non-compact) regular cover of a compact manifold \( N \) with metric lifted from \( N \), then \( \mathcal{M}^p(M) \) does not depend on the metric. One is led to expect in this case that \( \mathcal{M}^p(M) \) is a topological invariant of the pair \((N = M/\Gamma, \Gamma)\) in this case. In fact, Dodziuk [11] has shown that the action of \( \Gamma \) on \( \mathcal{M}^p(M) \) is a homotopy invariant of \((N, \Gamma)\) (up to equivalence). In particular, the \( L^2 \) Betti numbers \( \beta_i^p(N) = \dim_{\mathbb{R}} \mathcal{M}^p(M) \), c.f. §2, are homotopy invariants of \( N \).

In general, one is interested in understanding relations between the topology and geometry of \( M \) and the spaces \( \mathcal{M}^p(M) \). However, in many cases \( \dim \mathcal{M}^p(M) \) has been difficult to estimate, even whether it vanishes or not. Some examples and discussion follow.

[1.2] (1) \( \mathcal{M}^0(M) = \mathcal{M}^1(M) = \begin{cases} 0 & \text{if } \text{vol } M = \infty \\ \mathbb{R} & \text{if } \text{vol } M < \infty \end{cases} \). Further, if \( M \) is simply