A polynomial algorithm for partitioning line-graphs

Claudio Arbib
Dipartimento di Ingegneria Elettronica, Università di Roma "Tor Vergata"
v. O. Raimondo - 00173 Roma, Italy

Abstract

Two NP-complete unweighted graph partitioning problems are considered: Simple Max Partitioning (SMP) and Uniform Graph Partitioning (UGP). For both problems, polynomial-time algorithms are available for special classes of graphs. In the present paper, the class of line-graphs is considered and a polynomial algorithm is proposed to solve both UGP and SMP in this class.

1. Notations and preliminary definitions

In the following, a graph will be denoted by a pair (V,E), where V represents the set of vertices and E the set of edges. We also denote by V(G) and E(G) the set of vertices and the set of edges of G, respectively. If necessary, the cardinality of V(G) appears as a subscript, for example Vn. A complete graph with n vertices (clique) is denoted by Kn; a cycle is denoted by Cn. If (V,E) is a graph and x∈V, we set S(x) = (\{x\},E'), with E' = {eeE I e=(x,y)}.

The term graph partition indicates a partition of the vertex-set of a graph. Any graph partition \( <V_1,V_2> \) defines a cut \( C \), which is the subset of edges in E having one extreme in \( V_1 \) and the other one in \( V_2 \). The cardinality of the cut associated with \( p \) will be denoted by \( |p| \). A graph partition \( p \) is maximum if \( |p| \geq |q| \) for any graph partition \( q \).

If \( G=(V,E) \) is a graph, \( p=<V_1,V_2> \) a partition of \( V \) and \( G'=(V',E') \) a subgraph of \( G \), we say that \( p \) induces on \( G \) the partition \( p'=<V'\cap V_1,V'\cap V_2> \) and we write \( p' = p|G' \). If \( \phi(V) \) is the set of all the partitions \( <V_1,V_2> \) of \( V \) into two subsets, let us define a function \( \beta: \phi(V) \to \mathbb{Z}^+ \) by \( \beta(<V_1,V_2>) = |V_1| - |V_2| \). A partition \( p \) of \( V \) will be called uniform if \( \beta(p) \leq 1 \).

2. Two Unweighted graph partitioning problems

On an unweighted graph, the UNIFORM GRAPH PARTITIONING problem (UGP) can be formulated as follows:

Given a symmetric graph \( G=(V,E) \), find a uniform partition of \( V \) having maximum cardinality.
Such a graph partition is called \textit{UGP-optimal}. Any uniform partition of $V$ is called \textit{UGP-feasible}. If we relax the uniformity constraint, we get the problem of \textsc{Simple Max Partitioning (SMP)}\textsuperscript{1}.

The Uniform Graph Partitioning problem is NP-complete: Garey, Johnson and Stockmeyer proved in fact [6] that UGP can be polynomially reduced from Simple Max Cut (i.e. given a graph $(V,E)$, $s,t \in V$ and a positive integer $k$, find a partition $p$ of $V$ separating $s$ from $t$ and such that $|p| \geq k$), which is in turn as difficult as 3-Satisfiability. As for the complexity of SMP, it is easy to see that

\begin{align*}
2.1 & \textit{- THEOREM: Simple Max Partitioning is NP-complete.} \\

\text{} & \text{} \\

\text{} & \text{} \\

3. Preliminary results

In this Section we shall give some preliminary results.

3.1 - \textit{LEMMA: For any partition $p$ of $K_n$ we have}

\begin{align*}
(3.1.1) \quad |p| &= \frac{1}{4} \left( n^2 - \beta^2(p) \right) \\

\end{align*}

As a consequence, we have $|p| > |q|$ on a clique if and only if $\beta(p) > \beta(q)$. In particular, we immediately see that any uniform partition of a clique is both SMP- and UGP-optimal. Furthermore

3.2 - \textit{THEOREM: A graph $G$ is a clique if and only if all the uniform partitions of $G$ are SMP-optimal.} \\

Observe that the "only if" part follows from Lemma 3.1. The "if" part can be easily proved by induction (see [1]).

3.3 - \textit{LEMMA: Let $G$ be a graph and $\Gamma = \{G^1,\ldots,G^k\}$ a set of $k$ subgraphs of $G$ such that}

\begin{align*}
(i) & \quad V(G) = \bigcup_{i=1}^{k} V(G^i) \\
(ii) & \quad E(G) = \bigcup_{i=1}^{k} E(G^i) \\

\end{align*}

\text{} & \text{} \\

\text{} & \text{} \\

\footnote{\text{Other Authors (see e.g. [5,6,8]) prefer to state UGP as a minimization problem. In this case, however, the notion of disconnecting set instead of cut is more indicated.}}