MINIMAL IMMERSIONS OF $\mathbb{R}P^2$ INTO $\mathbb{C}P^n$
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In this paper(†) we consider minimal immersions of the real projective plane $\mathbb{R}P^2$ into the complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature 4. Since $\mathbb{R}P^n$ with its standard metric of constant curvature 1 sits in $\mathbb{C}P^n$ as a totally geodesic submanifold this includes the case of minimal immersions of $\mathbb{R}P^2$ into $\mathbb{R}P^n$. It is clear that all the immersions under consideration can be studied in the context of minimal immersions of $\mathbb{S}^2$ into $\mathbb{S}^n$ (with its standard metric) and $\mathbb{C}P^n$ with certain symmetry properties and this is what we shall do. For questions concerning the liftings of minimal immersions of $\mathbb{R}P^2$ into $\mathbb{R}P^n$ to minimal immersions of $\mathbb{S}^2$ into $\mathbb{S}^n$ we refer the reader to [11].

The motivation for considering such maps is two-fold:
(a) This class of immersions contains the most beautiful and most studied immersions of $\mathbb{S}^2$ into $\mathbb{S}^n$, namely the Veronese immersions, which are characterised by having induced metrics of constant curvature. Indeed our initial motivation was to try to prove the Simon conjecture [15] (which seeks to characterise the Veronese immersions by a pinching condition on the curvature of the induced metric) for this special class of immersions.
(b) An important aspect in the study of minimal immersions (and more generally harmonic maps) of $\mathbb{S}^2$ into $\mathbb{S}^n$ is the role of the higher order singularities in the sense of Chern [8]. (See also [4]). These are rather complicated in general but for the class of immersions considered here the singularity pattern determines the immersion up to ambient isometries almost completely. There is good reason to hope that the approach initiated here will lead to a much better understanding of the space of all minimal immersions (or more generally harmonic maps) of $\mathbb{S}^2$ into $\mathbb{S}^n$. The case $n = 4$ has been studied by Loo [14] and Verdier [16].

The approach we adopt is that initiated by Calabi [7] for minimal immersions of $\mathbb{S}^2$ into $\mathbb{S}^n$, and extended by others [9, 10, 12] to the case of minimal immersions of $\mathbb{S}^2$ into $\mathbb{C}P^n$, of relating the given immersion to a corresponding holomorphic curve called its directrix curve. The outline

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of the paper is as follows. In §1 we consider some basic facts about holomorphic curves in $\mathbb{C}P^n$ and their higher order singularity types including two key theorems. Then in §2 we recall the relationship between minimal immersions of $S^2$ into $\mathbb{C}P^n$ and holomorphic curves. In §3 we then deal with minimal immersions of $\mathbb{R}P^2$ into $\mathbb{C}P^n$ and state the main results. These are strong congruence results and in certain cases give explicit formulae for such immersions.

Readers unfamiliar with holomorphic curves may find [13] a useful reference for this material.

§1 Holomorphic curves

Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full holomorphic curve, where $S^2$ denotes the Riemann sphere. (We recall that a map $\psi : S^2 \to \mathbb{C}P^n$ is linearly full if there is no proper linear subspace $V$ of $\mathbb{C}^{n+1}$ such that $\psi(S^2) \subseteq P(V)$). We will say that $\psi_o$ has a higher order singularity of type $(u_1, \ldots, u_n)$ at $x \in S^2$ if for each $p = 1, \ldots, n$ the $(p-1)^{st}$ osculating curve [13] of $\psi_o$ has a singularity of index $u_p$ at $x$. In terms of a local complex coordinate $z$ on $S^2$ we may write $\psi(z) = [f(z)]$, where $f(z)$ is a non-vanishing $\mathbb{C}^{n+1}$-valued polynomial function and $[f(z)]$ denotes the line determined by $f(z)$. If $f^{(p)}$ denotes the $p$-th derivative of $f$ with respect to $z$ then $f, f^{(1)}, \ldots, f^{(n)}$ are linearly independent at all but a finite set of points, namely the set $Z(\psi)$ of points where the higher order singularities of $\psi_o$ occur. If the set $Z(\psi)$, possibly including $\infty$, has cardinality $k$ we say that $\psi$ is $k$-point ramified. Assume $\psi_o$ has a higher order singularity at $z = a$ of type $(u_1, \ldots, u_n)$. If, for $p = 1, \ldots, n$, we put $r_p = pu_1 + (p-1)u_2 + \ldots + u_p$ then

$$f \wedge f^{(1)} \wedge \ldots \wedge f^{(p)}(z) = (z-a)^{r_p} w_p(z)$$

for some $w_p(z)$ with $w_p(a) \neq 0$. In particular, $Z(\psi)$ is a finite set and, if $Z(\psi) = \{a_1, \ldots, a_k\}$, we call the set

$$\{(a_j ; u_1, (a_j), \ldots, u_n(a_j))) j = 1, \ldots, k$$

the singularity type of $\psi$. Note that projectively equivalent holomorphic curves have the same singularity type. The converse is false but as we shall see below a holomorphic curve is determined (modulo projective equivalence) up to a finite number of possibilities by its singularity type.