CHAPTER 6.

LINEAR PERIODIC SYSTEMS

6.1. Periodic families of bounded linear operators. Let $X$ be a Banach space, and let $T(t,s) : X \to X$, $t \geq s$, be a family of bounded linear operators satisfying the following properties:

(i) $T(s,s) = I$, the identity operator on $X$, for all $s \in \mathbb{R}$;
(ii) $T(t,u)T(u,s) = T(t,s)$ for all $t \geq u \geq s$;
(iii) there exists a constant $\omega > 0$ such that for any $t \geq s$,

\[ T(t+\omega,s+\omega) = T(t,s); \]

(iv) there exists a constant $\theta > 0$ such that $|T(t,s)| \leq \theta$

for all $0 \leq s \leq \omega$ and $s \leq t \leq s+\omega$.

The above family $T(t,s)$, $t \geq s$, is called an $\omega$-periodic family of bounded linear operators on $X$, or an $\omega$-periodic family on $X$, cf. Stech [1].

For any $s \in \mathbb{R}$, set

\[ P(s) = T(s+\omega,s). \]

$P(s)$ is called the period map. From Properties (ii) and (iii), it follows that $P^k(s) = T(s+k\omega,s)$, and hence

\[ (1.1) \quad T(t,s)P^k(s) = P^k(t)T(t,s) \]

for all $t \geq s$ and $k = 1, 2, \ldots$. Moreover, $P(s)$ is $\omega$-periodic:

$P(s+\omega) = P(s)$ for all $s \in \mathbb{R}$.

Now, let $\mu \neq 0$ be an eigenvalue of $P(s)$, and $\varphi$ be an eigenvector associated with $\mu$; $P(s)\varphi = \mu\varphi$. Then, (1.1) implies that $[P(t)-\mu I]T(t,s)\varphi = T(t,s)[P(s)-\mu I]\varphi = 0$ for $t \geq s$. Note that $T(t,s)\varphi \neq 0$ for $t \geq s$. Indeed, if $T(t,s)\varphi = 0$, then we may choose a nonnegative integer $m$ so that $s + m\omega \geq t$, and show that $0 = T(s+m\omega,t)T(t,s)\varphi = T(s+m\omega,s)\varphi = P^m(s)\varphi = \mu^m\varphi$. This contradicts the fact that both $\mu$ and $\varphi$ are nonzero. Thus, if $t \geq s$, $T(t,s)$ maps eigenvectors of $P(s)$ into eigenvectors of $P(t)$, and any nonzero eigenvalue of $P(s)$ is also an eigenvalue of $P(t)$. Conversely, if
μ is a nonzero eigenvalue of P(t), then it is also an eigenvalue of
P(s), because μ is an eigenvalue of P(s + mω) = P(s) for all m
such that s + mω > t as shown above. Thus, the nonzero point
spectrum of P(s) is independent of s ∈ R, and the null spaces
N(P(s) - μI), s ∈ R, are all of the same (perhaps infinite) dimension.

A complex number μ ≠ 0 is said to be a characteristic
multiplier of the ω-periodic family of T(t,s), if it is a normal
eigenvalue of P(s) for all s. A complex number λ for which e^{λω}
is a characteristic multiplier of T(t,s) is called a characteristic
exponent of T(t,s).

Let μ be a characteristic multiplier of T(t,s), and
{ψ₁, · · · , ψ_d} be a basis for the generalized eigenspace N((P(0) - μI)^K).
Define the basis vector Φ = (ψ₁, · · · , ψ_d).

Theorem 1.1 (Stech [1]). Let T(t,s), μ, k and Φ be as above.
Then, there exist a d × d constant matrix G and a d-row vector
Φ(t) in X such that σ(e^{Gω}) = {μ}, Φ(0) = Φ and Φ(t+ω) = Φ(t)
for all t ∈ R, and

(1.2) T(t,s)Φ(s) = Φ(t)e^{G(t-s)} for t ≥ s.

If b is any d-vector, then T(t,s)Φ(s)b can be defined for all t ∈ R by the relation

T(t,s)Φ(s)b = Φ(t)e^{G(t-s)}b.

Moreover, the generalized eigenspace of P(t) for μ is of the form
N((P(t) - μI)^K) for the common number k, its dimension is independent
t ∈ R, and Φ(t) defines a basis of N((P(0) - μI)^K).

Proof. Since μ is a normal eigenvalue of P(0), we can obtain
the decomposition of X as X = Y ⊕ R with σ(P(0)|_Y) = {μ} and
σ(P(0)|_R) = σ(P(0)) \ {μ} by Corollary 2.2 in Appendix, where
\( Y = N((P(0) - μI)^K) \) and \( R = \mathbb{R}((P(0) - μI)^K) \). Since P(0)Y ⊂ Y, there is a
d × d matrix M such that P(0)Φ = ΦM. The spectrum of M is
exactly \{μ\}, because σ(P(0)|_Y) = \{μ\}. Thus, there exists a d × d
matrix G such that M = e^{Gω}. Set Φ(t) = T(t,0)Φe^{-Gt} for t ≥ 0. Then

\[ Φ(t+ω) = T(t+ω,0)Φe^{-G(t+ω)} = T(t,0)T(ω,0)Φe^{-Gω} e^{-Gt} \]