HIGHER COHOMOLOGY OPERATIONS
THAT DETECT HOMOTOPY CLASSES

Zhou Xueguang
Nankai University,
Tianjin, China.

Let \( p \geq 5 \) be a prime, R.Cohen had proved in \([3]\) that \( b_k \otimes h_0 \) is an infinite cycle and represents a nontrivial element \( \xi_k \) of order \( p \) of the stable homotopy groups of sphere. R.Cohen and P.Goerss proved in \([4]\) that \( h_k \otimes h_0 \) is an infinite cycle and represents a nontrivial element \( \eta_k \) of order \( p \) for \( k \geq 2 \). Thus they obtained a kind of secondary cohomology operations and a kind of thirdary cohomology operations which detect homotopy classes. The main purpose of this paper is to obtain more kinds of higher cohomology operations that detect homotopy classes and then obtain more new infinite families of the stable homotopy groups of sphere. We prove that certain Toda's secondary product of \( \eta_2, \eta_3, \ldots, \xi_1, \xi_2, \ldots \) and \( p \), certain product of \( \beta_1, \beta_2, \eta_2, \eta_3, \ldots, \xi_1, \xi_2, \ldots, \alpha_1 \) are nonzero and we obtain their representation in the Adams spectral sequence, where \( \alpha_\xi \) and \( \beta_\xi \) denote respectively \( \alpha \) family and \( \beta \) family.

The main results of the present paper are the following:

Let \( R = \{k_1, k_2, \ldots, k_s, \xi_1, \xi_2, \ldots, \xi_s, j\} \), \( s \geq 0 \) be a set of \( 3s+1 \) nonnegative integers satisfying the following conditions:

1. \( k_i > k_{i+1} + 1 \), \( s - 1 \leq i \leq 1 \) and \( k_s \geq 2 \).
2. \( \xi_1 \leq 1 \), \( s \leq 1 \).
3. \( \xi_1 \leq 1 \), \( s \leq 1 \).
4. \( \xi_1 \leq 1 + j + 1 \).
5. \( \xi_1 \leq 1 + j + 1 \).

then we say that \( R \) is an \( H \) set and we put \( \ell(R) = \Sigma(2\xi_1 + \xi_s) + j + 1 \), \( r(R) = (2\xi_1 + \xi_s) + 1 \), where \( q = 2(p - 1) \) and sometimes we write \( j \) as \( j(R) \). We use \( \alpha \) to denote the expression

\[
\tau_1 \otimes \tau_1 \otimes \cdots \otimes \tau_1 \otimes h_k \otimes h_{k-1} \otimes \cdots \otimes h_1 \otimes h_0
\]

\( j \) copies
where \((\ )^d\) means that \((\ )\) appears \(d\) times. We use also \((\overline{A}_R) \in \text{ext}^{\ell}(R), r(R)(Z_p, Z_p)\) to denote the cohomology class containing \(\overline{A}_R\). For the meaning of \(\overline{A}_R\), see §1 and §4-6 in Part I, then we have

**Theorem 1:** Let \(p \geq 5\) be a prime, \(R\) be an \(H\) set, then

(a) \((\overline{A}_R) \neq 0\).

(b) \((\overline{A}_R)\) is an infinite cycle in the Adams spectral sequence and represents a nontrivial element \(P(R)\) of order \(p\) of the stable homotopy groups of spheres. If \(\lambda(S) = \lambda(R), r(S) = r(R)\), all the \((\overline{A}_S)\) constitute a basis of \(\text{Ext}^{\ell}(R), r(R)(Z_p, Z_p)\) and all the \(P(S)\) are linear independent in \(\pi_*(S^0)\).

Let \(D = \{\lambda, k, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}\) be a set of six nonnegative integers satisfying

1. \(\varepsilon_3 + 2(\lambda + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \geq p\).
2. \(\varepsilon_1 \leq 1, 1 \leq \varepsilon_2 \leq 4\).
3. \(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 1\)
4. \(k \geq 2\).

Then we say that \(D\) is a \(G\) set, we use \(B_D\) to denote the cohomology class

\[
(\varepsilon_2 \otimes h_1) \varepsilon_1 \otimes (h_k \otimes h_0)^{\varepsilon_2} \otimes (b_{k-1} \otimes h_0)^{\varepsilon_3} \otimes h_0^{\varepsilon_4} \otimes b_0^d
\]

and we put

\[
\lambda(D) = 2(\lambda + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4), \quad r(D) = q(k(\varepsilon_2 + \varepsilon_3) + p(2\varepsilon_1 + \lambda) + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)),
\]

then we have

**Theorem 2:** Let \(D\) be a \(G\) set, then

(a) \(B_D \neq 0\) in \(\text{Ext}^{\lambda}(D), r(D)(Z_p, Z_p)\).

(b) \(\beta_1^{\varepsilon_1} \beta_1^{\varepsilon_2} \beta_1^{\varepsilon_3} \beta_1^{\varepsilon_4} \neq 0\) in \(\pi_*(S^0)\).

In order to prove the main results, we need some new information of Ext groups of Steenrod algebra. In Part I, we use order cochain complex to reformulate Ext group of Hopf algebra. We prove that the \(E_2\) term of May spectral sequence inherit a coboundary operation from the cobar construction and Ext is the homology group with respect to the coboundary operation. Thus we get a simple cochain complex which is homology equivalent to the cobar construction. For the Adams spectral sequence which relating the homology group of Hopf algebra \(G, H\) and \(G/G\cdot H\), the cobar construction is also homology equivalent to \(H^{*\cdot*}(G/G\cdot H) \otimes H^{*\cdot*}(H)\), our results give an effective step to calculate the higher coboundary operations of these spectral sequence.

In Part II, we give some properties of Massey product and Toda's secondary product from \(\eta_k, \ldots, \xi_j, \ldots, p\). Using these results, we give the proof of theorem 1.