A QUANTUM STOCHASTIC CALCULUS IN FOCK SPACE OF INPUT AND OUTPUT NONDEMOLITION PROCESSES

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In this paper we introduce a new definition of the basic noncommutative measure for infinite-dimensional quantum noise in Fock space as the operator representation of a matrix *-Lie algebra of a pseudo-Euclidean space with indefinite metric \( \langle \xi | \xi \rangle = \xi^* - \xi^2 + |\xi|^2 + \xi^* \xi - \xi^2 \). In contrary to [1] we define the quantum stochastic integral as a uniformly continuous operator on a projective limit of Fock spaces. The described representation of the noncommutative calculus is closely connected with kernel calculus [2], but it is given directly in terms of operators instead of its kernels. The advantage of the quantum stochastic calculus gave us the possibility to prove [3] the main filtering theorem for the general output process, described by a quantum stochastic integral in Fock space.

1. A pseudo-Poissonian quantum process

Let \( \xi \) be a Hilbert integral \( \xi = \int_0^T \xi(t) \, dt \) of square-integrable vector-functions \( t \in \mathbb{R}^+ \mapsto \xi(t) \in \mathcal{H}(t) \) with a scalar product

\[
\langle \xi | \eta \rangle = \int_0^T \langle \xi(t) | \eta(t) \rangle \, dt = \int_0^T \xi^*(t) \eta^0(t) \, dt, \quad (1.1)
\]
where $\langle \xi | \eta \rangle(t) = \xi^* \eta(t)$ is the product in a vector space $\mathcal{E}(t)$, $\xi^0 = (\xi^1)$ is a column, representing $\xi(t)$ in a basis of $\mathcal{E}(t)$, and $\xi^*_o = (\xi^*_1)$ is the conjugated row, defined by $\xi^*_1 = \overline{\xi^1}$, $i \in J$. One can assume that all $\mathcal{E}(t)$ are equivalent to a Hilbert space $\mathcal{H}$ such that $\mathcal{E}$ can be identified with Hilbert tensor product $\mathcal{H} \otimes L^2(\mathbb{R}^+)$, in particular, $\mathcal{E} = L^2(\mathbb{R}^+)$ in the one-dimensional case $\mathcal{H} = \mathbb{C}$.

We shall denote by $\mathcal{F} = \mathcal{G}(\mathcal{E})$ the Fock space over $\mathcal{E}$, identified with the Hilbert integral $\mathcal{F} = \int \mathcal{G}(t) dt$ of square-integrable tensor-functions on $\tau \in \Omega^+ \mapsto \xi(\tau) \in \mathcal{E}(\tau)$, $\mathcal{E}(\tau) = \bigotimes_{t \in \tau} \mathcal{E}(t)$, where $\tau = (t_1, \ldots, t_n)$ is a chain $0 < t_1 < \ldots < t_n < \infty$ of finite length $n = |\tau| < \infty$ with natural Lebesgue measure $d\tau = dt_1 \ldots dt_n$. The Fock space scalar product

$$\int_0^\infty \langle \xi(\tau) | \eta(\tau) \rangle d\tau = \sum_{n=0}^\infty \int_0^\infty \int_0^\infty \cdots \int_0^\infty \langle \xi | \eta \rangle(t_1, \ldots, t_n) dt_1 \ldots dt_n$$

is defined by the products

$$\langle \xi | \eta \rangle(t_1, \ldots, t_n) = \xi^*_o(t_1, \ldots, t_n) \eta^0(t_1, \ldots, t_n), \xi^*_0 \eta = \xi^*_1 \ldots \xi^*_n \eta^* \eta^n$$

of tensors $\xi^*_o = (\xi^*_1 \ldots \xi^*_n)$, $\xi^*_1 \ldots \xi^*_n = \overline{\xi^1 \ldots \xi^n}$, conjugated to $\xi^0 = (\xi^0_1 \ldots \xi^0_n)$, which are the scalars $\xi^0(t_1, \ldots, t_n) \in \mathbb{C}$ in the case of $\mathcal{F} = L^2(\mathbb{R}^+)$, corresponding to $\mathcal{H} = \mathbb{C}$.

Due to the one-to-one correspondence of the finite chain set $\Omega^+$ and the Boolean ring of all finite subsets of $\mathbb{R}^+$, one can define on $\Omega^+$ the Boolean operations $\cap$, $\cup$, and difference $\tau \setminus \sigma = \tau \cap \overline{\sigma}$. We shall denote by $\emptyset$ the empty chain, and by $\xi^0 = \xi^\emptyset$ the vacuum function $\xi^\emptyset(\tau) = 0$, if $\tau \neq \emptyset$, $\xi^\emptyset(\emptyset) = 1$. According to the direct product representation $\Omega^+ = \bigotimes_{s} \Omega^+_s$ of any chain $\tau \in \Omega^+$ by the pair $\tau = (\tau^s, \tau^s)$, of $\tau^s = \{ t \in \tau | t \leq s \}$ and $\tau^s = \{ t \in \tau | t > s \}$, so that $\xi(\tau) = \xi(\tau^s) \otimes \xi(\tau^s)$, one can