A continuous time version of Stinespring's theorem on completely positive maps

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The aim of this note is to prove the following version of Stinespring's theorem for a one parameter semigroup of completely positive maps [1], [2].

Theorem: Let $A$ be $W^*$ algebra of operators containing the identity and acting in a Hilbert space $H_0$ and let $T_t : A \to A$ be a one parameter semigroup of completely positive linear maps such that $T_0$ is identity and $T_t(1) = 1$. Then there exists a family $\{H_t, t \geq 0\}$ of Hilbert spaces, $W^*$ homomorphisms $J_t : A \to B(H_t)$, the algebra of all operators on $H_t$ and isometries $V(s,t) : H_s \to H_t$, $0 < s < t < \infty$ such that

$$J_0(a) = a,$$

$$V(s,t) J_t(a) V(s,t)^* = J_s(T_{t-s}(a)),$$

$$V(t,u) V(s,t) = V(s,u)$$

for all $0 < s < t < u < \infty$.

Proof: For any Hilbert space $K$ we shall denote by $B(K)$ the $W^*$ algebra of all operators on $K$. For any $a \in B(K)$, $a^*$ will denote the adjoint of $a$. For any fixed $t$, applying Stinespring's theorem to the completely positive map $T_t$, construct the triple $(K_t, h_t, V_t)$ where $K_t$ is a Hilbert space, $h_t : A \to B(K_t)$ is a $W^*$ homomorphism and $V_t : H_0 \to K_t$ is an isometry such that

$$V_t^* h_t(a) V_t = T_t(a)$$

for all $a \in A$. 

and the set \{h_t(a)V_t u, a \in A, u \in H_t\} is total in $K_t$. We shall call

$(K_t, h_t, V_t)$ the Stinespring triple associated with $T_t$. Now, for

$0 < t_1 < t_2 < \infty$ consider the completely positive map $h_{t_1} \circ T_{t_2-t_1} : A \rightarrow B(K_{t_2-t_1})$

where $\circ$ denotes composition. Applying Stinespring's theorem to this

composed completely positive map construct the Stinespring triple

$(K_{t_1,t_2}, h_{t_1,t_2}, V_{t_1,t_2})$ where $K_{t_1,t_2}$ is a Hilbert space, $h_{t_1,t_2} : A \rightarrow B(K_{t_1,t_2})$

is a $W^*$ homomorphism and $V_{t_1,t_2} : K_{t_1,t_2} \rightarrow K_{t_1,t_2}$ is an isometry such that

$$V_{t_1,t_2}^+ h_{t_1,t_2}(a)V_{t_1,t_2} = h_{t_1} \circ T_{t_2-t_1}(a) \text{ for all } a \in A$$

and \{h_{t_1,t_2}(a)V_{t_1,t_2} u, a \in A, u \in K_{t_1,t_2}\} is total in $K_{t_1,t_2}$. Proceeding

inductively we obtain a family of Stinespring triples \{(K_{t_1}, h_{t_1}, V_{t_1}), 0 < t_1 < t_2 < \ldots < t_n < \infty, n = 1,2,\ldots\}$ where

$h_{t_1}, h_{t_1,t_2}, \ldots, h_{t_1,t_2,\ldots,t_n} : A \rightarrow B(K_{t_1,t_2,\ldots,t_n})$ is a $W^*$ homomorphism and $V_{t_1,t_2,\ldots,t_n} : K_{t_1,t_2,\ldots,t_n} \rightarrow K_{t_1,t_2,\ldots,t_n}$ is an isometry such that

$$V_{t_1,t_2,\ldots,t_n}^+ h_{t_1,t_2,\ldots,t_n}(a)V_{t_1,t_2,\ldots,t_n} = h_{t_1} \circ T_{t_2-t_1}(a) \text{ for all } a \in A$$

and \{h_{t_1,t_2,\ldots,t_n}(a)V_{t_1,t_2,\ldots,t_n} u, u \in K_{t_1,t_2,\ldots,t_n}\} is total in $K_{t_1,t_2,\ldots,t_n}$.

A map $X : [0, \infty) \rightarrow A$ is said to have finite support if the set

$S(X) = \{t : X(t) \neq 0\}$ is finite. Let $\bar{M}$ denote the set of all such maps

from $[0, \infty)$ into $A$ with finite support and let $\bar{M}_t \subset M$ consist of those elements

$X$ whose support $S(X)$ is contained in $[0,t)$. For any finite set

$F = \{t_1,t_2,\ldots,t_n\}, 0 < t_1 < t_2 < \ldots < t_n < \infty$ and $X \in M$ define

$$X(F) = h_{t_1,t_2,\ldots,t_n}(X(t_1))V_{t_1,t_2,\ldots,t_n}^+ h_{t_1,t_2,\ldots,t_n}(X(t_2))V_{t_1,t_2,\ldots,t_n}^+ \ldots$$

$$h_{t_1, t_2, \ldots, t_n}(X(t_n))V_{t_1, t_2, \ldots, t_n}^+ h_{t_1, t_2, \ldots, t_n}(X(t_1))V_{t_1, t_2, \ldots, t_n}$$

(2)