1. Introduction

Let $R$ be a (commutative, noetherian) ring and let $I \subseteq R$ be an ideal. The main object envisaged in this talk is the Rees algebra of $I$, namely, the graded $R$-algebra $R(I) = R \oplus I \oplus I^2 \oplus \ldots$. A related ring is the associated graded ring of $I$, to wit, $\text{gr}_I(R) := R(I)/IR(I)$. Roughly, the latter can be viewed as a specialization of the former and, in a vague sense, its description is closer to $R$ than that of $R(I)$. Also, historically perhaps, for reasons of analysing singularities and their resolutions, it was $\text{gr}_I(R)$ that played a central role. As arithmetical questions – in the sense algebraic geometers mean referring to properties of the underlying coordinate ring – increasingly plagued the scene, the emphasis began leaning towards a deeper understanding of the structure of $R(I)$.

Two broad questions, along with their variations, seem to be unanswered:

1. Find reasonable conditions under which $R(I)$ is normal.
2. Find natural obstructions to the Cohen-Macaulayness of $R(I)$.

By definition, the two phenomena intersect along the well-known property $S_2$ of Serre. Beside this obvious hunch that the properties ought to have more in common, there is no further general evidence to a stronger relationship between them, except for the one single result due to Hochster to the effect that normal subrings generated by monomials in indeterminates over a field are always Cohen-Macaulay. Despite such a lack of evidence, some of us still believe that in the cases of Rees algebras of ideals in special classes the overlapping ought to be detected in a more precise way.

Some of the results stated in this paper will appear in toto in [HSV 4]. Two substantial sections are entirely novel, to wit, Section 4, on Graphael ideals, and Section 5, on monomial ideals in two variables. The remaining parts contain variants of proofs of some results in [BST].

2. Normalitana

In this section, we describe a few general techniques for sorting out normal ideals. As a matter of notation, we will sometimes denote the Rees algebra of the ideal $I$ by $R[I(t)]$ to emphasize the embedding $R(I) \hookrightarrow R[t]$. In this line, yet another algebra that plays an important role in the theory is the so-called extended Rees algebra of the ideal $I$, defined as $R[I(t), t^{-1}] \subset R[t, t^{-1}]$.

A first criterion for normality is the following bunch of mutually equivalent statements.
(2.1) Proposition. Let \( R \) be a noetherian normal domain and let \( I \subset R \) be an ideal. The following conditions are equivalent:
(i) The Rees algebra \( R[It] \) is normal.
(ii) The ideal \( IR[It] \subset R[It] \) is integrally closed.
(iii) \( I \) is normal.
(iv) \( (R \text{ local with maximal ideal } m) \ I \text{ is locally normal in the punctured spectrum } \text{Spec} \ R \setminus \{m\} \) and there exists an element \( f \in mR[It] \) such that \( R[It]_P \) is normal for every \( P \in \text{Ass} R[It]/fR[It] \).
(v) \( \text{The ideal } (t^{-1}) \subset R[It, t^{-1}] \) is integrally closed.
(vi) \( R[It, t^{-1}]_P \) is normal for every \( P \in \text{Ass} \text{gr}_I(R) \).
(vii) \( R[It, t^{-1}] \) is normal.

The proof of the equivalences is easy (cf. [HSV 4]).

Application. Let \( I \) be the ideal generated by the \( n-1 \times n-1 \) minors of a generic \( n \times n \) matrix \( X \) (over a field \( k \)). We show the conditions in (iv) are presently met. First, there is no harm in localizing at \( (X) \). Change notation: \( R := k[X]_{(X)}, I := I_{(X)}, \) etc. By induction on \( n \), using the inversion and elementary transformation trick, we have that \( I \) is locally normal outside \( (X) \). Set \( f := \det(X) \). By Huneke's [Hu 2] the ideal \( (It) \subset R[It] \) is radical, hence we are done.

Another criterion bearing theoretical importance is given in terms of normally divisorial ideals.

(2.2) Definition. An ideal \( I \subset R \) is normally divisorial if the exceptional ideal \( IR[It] \) has a primary decomposition
\[
IR[It] = P_1^{(l_1)} \cap \cdots \cap P_r^{(l_r)},
\]
where \( P_i \) is a height one prime ideal of \( R[It] \) and \( P_i^{(l_i)} \) stands for its \( l_i \)-th symbolic power.

(2.3) Proposition. Let \( R \) be a normal domain and let \( I \subset R \) be an ideal. The following conditions are equivalent:
(i) \( I \) is normally divisorial.
(ii) \( I \) is normal.

The implication (i) \( \Rightarrow \) (ii) is certainly well-known. The reverse one is the bulk of the Proposition: it can be proven by a technique of passing to the extended Rees algebra of \( I \) and, once there, applying reduction theory in one-dimensional local rings. For the details, refer to [HSV 4].

In a different vein, there is also a computationally-oriented version of Serre's criterion. Namely, let \( J \) stand for a presentation ideal of \( R(I) \), i.e., the kernel of a surjective \( R \)-homomorphism
\[
R[T] := R[T_1, \ldots, T_n] \twoheadrightarrow R[It]
\]
defined by the assignment \( T_i \mapsto x_i t \) for a given set of generators \( x_1, \ldots, x_n \) of \( I \). Then, one can state:

(2.4) Proposition. ([BSV]) Let \( R \) be a normal domain and let \( I \subset R \) be an ideal. The following conditions are equivalent:
(i) \( R[It] \) is normal.
(ii) The following statements hold:
(1) The ideal \( (I, J)R[T] \) has no embedded primes;