ON SOME PROBLEMS FROM THE THEORY OF FIXED POINTS OF MULTIVALUED MAPPINGS

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In the theory of fixed points of multivalued mappings (M-mappings) two different approaches can be easily traced. The approximation approach which stems from the paper [I] is based on the substitution of a M-mapping by a close to it (in some sense) single-valued mapping. The homological approach going back to [2] is based on the studying of the homological structure of the graph of a M-mapping.

This paper is devoted to the studying of fixed points of M-mappings with arbitrary compact images in finite dimensional spaces. The topological characteristic is studied which is an invariant of a M-mapping that enables to prove some new theorems about fixed points. This invariant also establishes a correspondence between homological and approximation approaches.

The notion of a limit value of a M-mapping at a point is also considered and used to study fixed points.

The results of this paper adjoin the author's papers [3, 4] and the surveys [5, 6].

I. A topological characteristic of M-mappings.

I.1. Main definitions and theorems. Let X be a topological space. Denote by C(X) (K(X)) the set of all non-empty closed (compact) subsets of X.

Let Y ⊂ X. A point x₀ ∈ Y is called fixed point of a M-mapping \( F : Y \longrightarrow C(X) \) if \( x₀ \subseteq F(x) \). Denote by the set of fixed points of a M-mapping \( \text{Fix}(F) \).

It is known that the existence of fixed points of the M-mapping
substantially depend from the structure of images of this mapping.

The following is proven in [7].

Theorem I. If Hausdorff topological space $X$ is arcwise connected and locally contractible, then the homotopic groups of the set $K(X)$ equipped with exponential topology are trivial.

This statement shows that topological invariant (a kind of topological degree) which is stable under homotopies cannot be constructed even in the class of continuous $M$-mappings with arbitrary compact images. However, in order to study fixed points of such $M$-mappings one may use the ideas based on the notion of homological obstruction to the extension of continuous mappings. We will present this construction following [3, 5].

Let $E$ be a vector space, $X \subset E$, and $F : X \to C(E)$ be some $M$-mapping. Consider the graph $\Gamma_X(F)$ of the $M$-mapping $F$ over the set $X$. Then the continuous mappings $r : \Gamma_X(F) \to E$, $t : \Gamma_X(F) \to X$ are defined, which are the restrictions on $\Gamma_X(F)$ of the natural projections $\rho^E : X \times E \to E$, $\rho_{X} : X \times E \to X$. Evidently, for each $x \in X$ we have an equality $F(x) = r \circ t^{-i}(x)$. Furthermore, multivalued vector field (MV-field) $\Phi = i - F$ where $i$ is the imbedding of $X$ into $E$ can be represented in the form $\Phi(x) = (t-r) \circ t^{-i}(x)$. Let us give a more general definition.

Definition I. A $M$-mapping $F : X \to K(E)$ is said to be determined by a quintuple $F = (X,E,Z,f,g)$ if $X,Y,Z$ are topological spaces, $f : Z \to X$ is a continuous surjective proper mapping, $g : Z \to E$ is a continuous mapping, and for each $x \in X$ the equality $F(x) = g \circ f^{-i}(x)$ holds. $(X,E, \Gamma_X(F), t,r)$ is an example of such quintuple. It will be called the canonical quintuple of a $M$-mapping $F$.

Let $U$ be a bounded region in a finite-dimensional Euclidian space $\mathbb{R}^{n+1}$, $\partial U = L$. Consider a $M$-mapping $F : \bar{U} \to C(\mathbb{R}^{n+1})$ which has no fixed points on $L$. Let $\Gamma_{\bar{U}}(F)$ and $\Gamma_{L}(F)$ be the graphs of the $M$-mapping $F$ over $\bar{U}$ and $L$ respectively. MV-field $\Phi = i-F;\bar{U} \to C(\mathbb{R}^{n+1})$ does not vanish on $L$, so that the mapping of pairs

$$t-r : (\Gamma_{\bar{U}}(F), \Gamma_{L}(F)) \to (\mathbb{R}^{n+1}, \mathbb{R}^{n+1}\setminus 0)$$

is defined, which induces a homomorphism $(t-r)^*_{n}$ of Alexandrov-

Čech cohomology groups of these spaces.

Consider the composition $\nu_{G} = \delta \circ (t-r)^*_{n}$ of homomor-