CHAPTER X

ESTIMATES ON THE V-FUNCTIONS

In this chapter we prove Theorem 9.2.1 studying an integral equation for \( v_n \) by an iterative procedure similar to that used to prove Theorem 5.4.3. Here however we have the extra difficulty of the exclusion interaction and also the case \( L_G \equiv 0 \) is non trivial. We proceed by steps considering first the stirring process alone and then the general case.

§10.1 The v-functions in the stirring process.

To have an easier comparison with the existing literature, we go back to the particles language, where the occupation numbers are \( \eta(x) = 0, 1 \). We realize the stirring process by means of the active-passive marks process (cf. §6.4.5). We set, for notational simplicity, \( E = 1 \): since the generator is \( \epsilon^{-2}L_0 \) the process depends on \( \epsilon \) and \( t \) via \( \epsilon^{-2}t \), hence the case \( \epsilon < 1 \) is immediately recovered. We denote by \( \rho(x,t) \) the solution of

\[
\frac{\partial \rho(x,t)}{\partial t} = \frac{1}{2} \left[ \rho(x+1,t) + \rho(x-1,t) - 2\rho(x,t) \right], \quad \rho(x,t) = \eta(x)
\]

and, by (2.11),

\[
\rho(x,t) = E_\eta(\eta(x,t)) \tag{10.1}
\]

For any integer \( n \) we set

\[
\mathcal{M}_n = \{ \underline{x} = (x_1, \cdots, x_n) : x_i \neq x_j \quad \forall i \neq j \} \tag{10.2}
\]

and for any \( \underline{x} \in \mathcal{M}_n \)

\[
v_n(\underline{x}, t|\eta) = E_\eta \left( \prod_{i=1}^{n} [\eta(x_i,t) - \rho(x_i,t)] \right), \quad v_n(\underline{x},0) = 0 \tag{10.3}
\]

In the remaining of this paragraph we give a proof of the following theorem, firstly proven in [62],

10.1.1 Theorem. There is a sequence \( c_n \) such that for all \( \eta \in \{0,1\}^\mathbb{Z} \) and all \( t > 0 \)

\[
\sup_{\underline{x} \in \mathcal{M}_n} |v_n(\underline{x},t|\eta)| \leq c_n t^{-n/8} \tag{10.4}
\]

In the sequel we shall simply write \( v_n(\underline{x},t) \) for \( v_n(\underline{x},t|\eta) \). We first establish an integral equation for the v-functions:
10.1.2 Lemma. For any $x \in \mathcal{M}_n$ and any $t \geq 0$,

$$v_n(x, t) = \int_0^t ds \sum_x P_x(x \rightarrow x)(Av)(x, t - s)$$  \hspace{1cm} (10.5)$$

where, for any $t > 0$, $P_t(x \rightarrow x)$ is the stirring process transition probability to go from $x$ to $x$ in a time $t$. $(Av)$ is given by

$$(Av)(x, t) = \sum_{h,k=1}^{n} 1(x_h = x_k + 1)\{[\rho(x_h, t) - \rho(x_k, t)][v_{n-1}(x^{(h)}, t) - v_{n-1}(x^{(k)}, t)] - \frac{1}{2} [\rho(x_h, t) - \rho(x_k, t)]^2 v_{n-2}(x^{(h,k)}, t)\}$$

where $x^{(h)}(t)$ and $x^{(k)}(t)$ are respectively equal to $\{x_i(t), i \neq h\}$, and $\{x_i(t), i \neq k\}$, while $x^{(h,k)}(t) = \{x_i(t), i \neq h, k\}$.

Proof. We have

$$\frac{d}{dt}v_n(x, t) = \mathbb{E}_x\left(\int_0^t ds L_0[v_n(x, t) - (Av)(x, t)]\right)$$

From this it is not difficult to check that

$$\frac{d}{dt}v_n(x, t) = L_0 v_n(x, t) + (Av)(x, t)$$

hence (10.5). $\square$

Denoting by $\mathcal{P}(x)$ the process of $n$ stirring particles, we can rewrite (10.5) as

$$v_n(x, t) = \int_0^t ds \mathbb{E}_x((Av)(x, t))$$

In the expression for $(Av)$ there are terms which are small: by the local central limit theorem, see (4.34), in fact, for any $t > 0$, and $z_i, z_j \in \mathbb{Z}$

$$|\rho(z_i, t) - \rho(z_j, t)| = |\sum_z [P_t(z_i \rightarrow z) - P_t(z_j \rightarrow z)] \eta(z)| \leq c' \frac{|z_i - z_j|}{1 + t^{1/2}} \leq c \frac{|z_i - z_j|}{t^{b+1/2}}$$

The last expression gives a worse estimate when $t \rightarrow \infty$, but it will allow for more compact notation in the sequel.

The bound (10.9) by itself is not sufficient for proving Theorem 10.1.1, we need to take into account also the characteristic function and the difference between the two $v$-functions both present in (10.6). We start by the latter.

10.1.3 Lemma. For any $x \in \mathcal{M}_n$, any $i \neq j$ in $\{1, \ldots, n\}$ and any $t > 0$:

$$v_{n-1}(x^{(j)}, t) - v_{n-1}(x^{(i)}, t)$$

$$= \int_0^t ds \mathbb{E}_x\left(1(\tau_{i,j} > s)[(Av)(x^{(j)}, t - s) - (Av)(x^{(i)}, t - s)]\right)$$

(10.10)