1 Introduction

The main purpose of this survey is to present and popularize the notion of a random dynamical system (RDS) and to give an impression of its scope. The notion of RDS covers the most important families of dynamical systems with randomness which are currently of interest. For instance, products of random maps — in particular products of random matrices — are RDS as well as (the solution flows of) stochastic and random ordinary and partial differential equations.

One of the basic results for RDS is the Multiplicative Ergodic Theorem (MET) of Oseledec [38]. Originally formulated for products of random matrices, it has been reformulated and reproved several times during the past twenty years. Basically, there are two classes of proofs. One makes use of Kingman's Subadditive Ergodic Theorem together with the polar decomposition of square matrices. The other one starts by proving the assertions of the MET for triangular systems, and then enlarges the probability space by the compact group of special orthogonal matrices, so that every matrix cocycle becomes homologous to a triangular one.

Let us emphasize that the MET is a linear result. It is possible to introduce Lyapunov exponent-like quantities for nonlinear systems directly à la (9) below, or, much more sophisticated, as by Kifer [27]. However, the wealth of structure provided by the MET is available for linear systems only. Speaking of an “MET for nonlinear systems” always means the MET for the linearization of a nonlinear system. What is new for nonlinear systems is the fact that the linearization lives on the tangent bundle of a manifold (instead of the flat bundle $\mathbb{R}^d \times \Omega$ as for products of random matrices). The MET yields nontrivial consequences for deterministic systems already. This case has been dealt with by Ruelle [39]. Ruelle's argument proceeds by trivialization of the nonflat tangent bundle. It is exactly the same argument that works for nonlinear random systems: infer the MET for the linearization of the system from the ordinary MET together with a trivialization argument. We reproduce the argument below.

Stochastic flows have entered the scene a couple of years ago. They are related to RDS, but they are not the same. We describe their relations, and point out their differences.

The final Section briefly reviews all contributions to the present volume.
2 Random Dynamical Systems and Multiplicative Ergodic Theory

2.1 RDS

Consider a set \( T \) (time), \( T = \mathbb{R}, \mathbb{Z}; \mathbb{R}^+, \text{or } \mathbb{N} \), and a family \( \{ \theta_t : \Omega \to \Omega \mid t \in T \} \) of measure preserving transformations of a probability space \( (\Omega, \mathcal{F}, P) \) such that \( (t, \omega) \mapsto \theta_t \omega \) is measurable, \( \{ \theta_t \mid t \in T \} \) is ergodic, and \( \theta_{t+s} = \theta_t \circ \theta_s \) for all \( t, s \in T \) with \( \theta_0 = \text{id} \). Thus \( (\theta_t)_{t \in T} \) is a flow if \( T = \mathbb{R} \) or \( \mathbb{Z} \), and a semi-flow if \( T = \mathbb{R}^+ \) or \( \mathbb{N} \). The set-up \( ((\Omega, \mathcal{F}, P), (\theta_t)_{t \in T}) \) is a (measurable) dynamical system.

Definition A random dynamical system on a measurable space \( (X, \mathcal{B}) \) over \( (\theta_t)_{t \in T} \) on \( (\Omega, \mathcal{F}, P) \) is a measurable map 
\[
\varphi : T \times X \times \Omega \to X
\]
such that \( \varphi(0, \omega) = \text{id} \) (identity on \( X \)) and
\[
\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)
\]
for all \( t, s \in T \) and for all \( \omega \) outside a \( P \)-nullset, where \( \varphi(t, \omega) : X \to X \) is the map which arises when \( t \in T \) and \( \omega \in \Omega \) are fixed, and \( \circ \) means composition. A family of maps \( \varphi(t, \omega) \) satisfying (1) is called a cocycle, and (1) is the cocycle property.

We often omit mentioning \( ((\Omega, \mathcal{F}, P), (\theta_t)_{t \in T}) \) in the following, speaking of a random dynamical system (abbreviated RDS) \( \varphi \).

We do not assume the maps \( \varphi(t, \omega) \) to be invertible a priori. By the cocycle property, \( \varphi(t, \omega) \) is automatically invertible (for all \( t \in T \) and for \( P \)-almost all \( \omega \)) if \( T = \mathbb{R} \) or \( \mathbb{Z} \), and \( \varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega) \).

The following examples are quite distinct in many respects. However, they all are RDS.

1. The simplest case of a random dynamical system is a non-random — viz., deterministic — dynamical system. An RDS is deterministic if \( \varphi \) does not depend on \( \omega \), i.e., \( \varphi(t, x, \omega) = \varphi(t, x) \). Then the cocycle property (1) reads \( \varphi(t+s) = \varphi(t) \circ \varphi(s) \), hence \( (\varphi(t))_{t \in T} \) consists of the iterates of a measurable map on \( X \) if \( T = \mathbb{Z}^+ \) and \( (\varphi(t))_{t \in T} \) is a measurable (semi-) flow if \( T = \mathbb{R}^+ \), respectively.

2. Let \( \theta : \Omega \to \Omega \) be a measure preserving transformation, and let \( \psi : X \times \Omega \to X \) be a measurable map. Put \( \psi_n = \psi \circ \theta^{n-1} \). Then
\[
\varphi(n, \omega) = \begin{cases} 
\psi_n(\omega) \circ \psi_{n-1}(\omega) \circ \ldots \circ \psi_1(\omega) & \text{for } n > 0 \\
\text{id} & \text{for } n = 0 \\
\psi^{-1}_{n+1}(\omega) \circ \psi^{-1}_{n+2}(\omega) \circ \ldots \circ \psi^{-1}_0(\omega) & \text{for } n < 0,
\end{cases}
\]
defines an RDS (of course, defining \( \varphi(n, \omega) \) for \( n < 0 \) needs \( \theta \) and \( \psi(\cdot, \omega) \) invertible \( P \)-a.s.). In particular, if \( X = \mathbb{R}^d \) and \( x \mapsto \psi(x, \omega) \) is linear, then \( \varphi \) is a product of random matrices.