A second order extension of Oseledets theorem

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Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\theta\) an invertible ergodic transformation.

Let \(E\) be a real vector space of dimension \(d\).

Let \(A\) be a measurable map from \(\Omega\) into the group of isomorphisms of \(E\). We assume that \(E(\log^+ \|A\| + \log^+ \|A^{-1}\|)\) is finite.

Set \(A_n(\omega) = A(\omega)A(\theta\omega)\ldots A(\theta^{n-1}\omega)\).

For \(p \in \{1,2,\ldots,n\}\), define \(s_p = \lim_{n \to \infty} \frac{1}{n} \log \|A_{n}^{p}\|\). Set \(\lambda_1 = s_1\) and \(s_p - s_{p-1} = \lambda_p\) for \(2 \leq p \leq d\).

We assume that \(\lambda_1\) is positive and that the Lyapounov exponents \(\lambda_p\) are distinct.

By Oseledets theorem (cf. [1]) we can find a random set of directions \(\tau_p(\omega), 1 \leq p \leq d\), such that \(-\lim_{n \to +\infty} \frac{1}{n} \log \|A_{n}^{-1}(\omega) \tau_p(\omega)\| = \lambda_p\) and

\[
\frac{A_{n}^{-1} \tau_p}{\|A_{n}^{-1} \tau_p\|} = \tau_p \circ \theta^n.
\]

Equivalently, we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \|A_{n}(\omega) \tau_p (\theta^n \omega)\| = \lambda_p\] and \(\frac{A(\tau_p \circ \theta^n)}{\|A(\tau_p \circ \theta^n)\|} = \tau_p\).

Let \(r, 1 \leq r < d\) be such that \(s_r > 0\).

Set \(\sigma_r = \tau_1 \wedge \tau_2 \ldots \wedge \tau_r\).

Let \(B\) be a random map from \(\Omega\) into \(\text{Hom}(E \wedge r, E \wedge r)\), such that \(E(\log^+ \|B\|)\) is finite.
Proposition.

a) The series \( R(\omega) \) converges absolutely \( P \) almost surely. \( R \) is determined by the identity:

\[
R = \frac{B \sigma_r}{\|\sigma_r\|^2} + \frac{A_{n}^r \tau(\theta \omega)}{\|A_{n}^r \sigma_r\|^2}.
\]

b) For any \( r \)-direction \( v \), such that \( v \neq \sigma_r \) a.s.,

\[
R^v_N(\omega) = \sum_{n=0}^{N} \frac{A_{n}^r(\omega) B(\theta \omega)(A_{n}^r(\theta \omega)v)^{\otimes 2}}{\|A_{n}^r(\omega)v\|^2}
\]

converges in probability towards \( R(\omega) \).

c) This convergence holds a.s. for Lebesgue almost all \( r \)-direction \( v \).

Proof: We take \( r = 1 \) for notational simplicity.

Fix \( 0 < \varepsilon < \frac{\lambda_1 - \lambda_2}{100} \).

Set \( H(\omega) = \sup_n \|B(\theta \omega)\| e^{-\varepsilon n} \) (finite by Borel-Cantelli's lemma) (1).

\[
D_p(\omega) = \sup_n \left( \frac{\|A_{n}^p(\omega)\|}{\|A_{n}^p\|} e^{-(s+\varepsilon)n} \right) \quad (2)
\]

\[
K_p(\omega) = \inf_n \left( \frac{\|A_{n}^\tau(\theta \omega)\|}{\|A_{n}^\tau\|} e^{-(\lambda - \varepsilon)n} \right) \quad (3)
\]

\[
Q_p(\omega) = \sup_n \left( \frac{\|A_{n}^\tau(\theta \omega)\|}{\|A_{n}^\tau\|} e^{-\varepsilon n} \right) \quad (4)
\]

a) follows from the majoration

\[
|R(\omega)| \leq \frac{\|A_{n}^p\| \|B_n\|}{\|A_{n}^\tau(\theta \omega)\|^2} \leq K^{-2} D_{1} H(\omega) \sum e^{(4\varepsilon - \lambda_1)n}
\]

To prove b) (c) we show that \( |R_N - R^v_N| \) converges in probability (a.s.) towards 0 where \( R_N \) is the sum of the \((N+1)\) first terms of \( R \).

For any vector \( v \in E \), let \( \sum_{i=1}^{d} \eta_{1i}(v) \) be its decomposition along the directions \( \tau_1(\omega) \).