Let $Z$ be a complex-valued process and let $X$ and $Y$ be its real and imaginary parts respectively, so that $Z = X + iY$. Let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing family of complete $\sigma$-fields. We say that $\{Z_t, \mathcal{F}_t, t \geq 0\}$ is a conformal martingale if both $X$ and $Y$ are continuous local martingales relative to $\{\mathcal{F}_t\}$, such that $<X,Y>_t \equiv 0$ and $<X,X>_t = <Y,Y>_t$. If the point $t=0$ is not included in the parameter set, we will say that $\{Z_t, \mathcal{F}_t, t>0\}$ is a conformal martingale if for all $\delta>0$, $\{Z_t, \mathcal{F}_t, t>\delta\}$ is a conformal martingale. We refer the reader to (1) for the properties of conformal martingales. We want to call attention to the following property which, though elementary, is still curious.

**Proposition:** Let $\{Z_t, \mathcal{F}_t, t>0\}$ be a conformal martingale. Then, for a.e. $\omega$, one of the following happens.

Either

(i) $\lim_{t \to 0} X_t(\omega)$ exists in the Riemann sphere,

or

(ii) for each $\delta>0$, $(X_t(\omega), 0<t<\delta)$ is dense in $\mathbb{C}$. 

Remark: Both possibilities can occur. Indeed, if $B_t$ is a complex Brownian motion from $0$ and if $f$ is holomorphic in $\mathbb{C} - \{0\}$, then $\{f(B_t), t > 0\}$ is a conformal martingale. If $0$ is a removable singularity, $\lim_{t \to 0} f(B_t)$ exists. If it is a pole, $\lim_{t \to 0} f(B_t) = \infty$, and if it is an essential singularity, $\{f(B_t), 0 < t < \delta\}$ is dense in $\mathbb{C}$ for each $\delta > 0$. Thus the above proposition is the analogue for conformal martingales of Weierstrass' theorem.

Proof: All we must show is that if (i) doesn't happen, (ii) does. The only fact about conformal martingales we will need is that if $\{Z_t, t > t_0\}$ is a conformal martingale, it can be time-changed into a complex Brownian motion with a possibly finite lifetime([1] or [2], p. 384). Thus all hitting probabilities for $Z$ are dominated by those of Brownian motion.

Suppose (i) doesn't happen. Then there exist concentric circles $C_1, C_2$ with a rational center $z_0$ and rational radii $r_1 < r_2$ respectively, such that the number of incrossings of $(C_1, C_2)$ by $Z_t(\omega)$ is infinite. Here, the number of incrossings $\nu_{a,b}(\omega)$ of $(C_1, C_2)$ in $(a,b)$ is defined to be the number of downcrossings (in the usual sense) of the interval $(r_1, r_2)$ by the process $\{|Z_t(\omega) - z_0|, a < t < b\}$. Let $D$ be a disc.

Suppose that $D$ is not entirely contained in the interior of $C_1$. (If it is, we merely talk about outcrossings rather than incrossings in what follows.) The proposition will be proved if we can show that for any $\delta > 0$, $T_D < \delta$ a.s. on the set $\{\nu_{0,\delta} = \infty\}$, where $T_D = \inf\{t > 0 : Z_t \in D\}$. 