A ring $S$ is called a (finite) normalizing extension of a ring $R$ if $R \subseteq S$ and $S = \bigoplus_{i=1}^{n} a_i R$ with $a_i R = R a_i$ for each $i$. Our aim in this paper and a sequel [8] is to extend to such extensions some of the results concerning the prime ideals of $R$ and $S$ obtained earlier for finite centralizing, or liberal, extensions by Bergman and by Robson and Small (see [6,7]) and, for crossed products $S = R \rtimes G$ with $G$ a finite group, by Lorenz and Passman [4]. These are, of course, both examples of normalizing extensions. Some of the methods used here are applied in another sequel [1] to study chain conditions in $R$ and $S$.

The basic technique is module-theoretic, viewing $S$ as an $R$-bimodule. Thus Section 1 discusses the relationship between $S$ and $R$ modules and bimodules. This is applied in Section 2 to give the structure of $I \cap R$, for $I$ a prime ideal of $S$, and to prove a "lying-over" theorem for prime ideals of $R$. Incomparability will be discussed in [8].

This paper has an overlap (namely, Corollary 2.2 and much of Theorem 2.1) with recent independent work of Lorenz [5]. Throughout this paper $R, S$ and $\{a_i | i = 1, \ldots, n\}$ will be fixed as described above; and $R$ and $S$ will share a common identity element.

1. MODULES AND BIMODULES

In this section we first discuss the underlying $R$-module structure of a right $S$ module $M$; then we comment on the situation for bimodules.

Suppose $N$ is an $R$ submodule of $M$. We let $Na_i^{-1} = \{m \in M | ma_i \notin N\}$. This is also an $R$ submodule.

**Lemma 1.1** The natural group monomorphism $M/Na_i^{-1} \to M/N$ given by $m + Na_i^{-1} \to ma_i + N$ induces a lattice embedding $\mathcal{L}(M/Na_i^{-1}) \to \mathcal{L}(M/N)$ of the lattices of $R$ submodules.

**Proof** This is easily verified.

We define $b(N)$, the bound of $N$, to be the largest $S$ submodule of $M$ contained in $N$. In fact $b(N) = \bigcap Na_i^{-1}$ as is easily checked.
LEMMA 1.2 If $N$ is an essential $R$ submodule of $M$ then so too is $b(N)$.

Proof Let $A$ be an $R$ submodule of $M$. If $Aa_i = 0$ then $A \subseteq Na_i^{-1}$. Otherwise $Aa_i \cap N \neq 0$ and so $A \cap Na_i^{-1} \neq 0$. Therefore $Na_i^{-1}$ is an essential $R$ submodule of $M$. Hence so too is $\bigcap Na_i^{-1} = b(N)$.

LEMMA 1.3 $M$ contains an $R$ submodule $N$ maximal with respect to $b(N) = 0$.

Proof Let $\{N_k \mid k \in I\}$ be a chain of $R$ submodules such that $b(N_k) = 0$ for each $k$. If $b(\bigcup N_k) \neq 0$ then $\bigcup N_k \nexists xS$ for some $0 \neq x \in \bigcup N_k$. Hence $xa_i \in \bigcup N_k$ and so $xa_i \in N_k$ for some $k = k(i)$. It follows that $xS \subseteq N_k$ for $k = \sup k(i)$, a contradiction to $b(N_k) = 0$. Hence $b(\bigcup N_k) = 0$ and so Zorn's lemma can be applied to give $N$.

We let $\text{rank}_R, \text{rank}_S$ denote uniform ranks of a module over these rings.

LEMMA 1.4 Let $\text{rank } M_S = m$ and let $N$ be an $R$ submodule maximal with respect to $b(N) = 0$. Then $\text{rank}(M/N)_R \leq m$ and $\text{rank } M_R \leq mn$.

Proof Let $A_1, \ldots, A_t$ be $R$ submodules of $M$ strictly containing $N$ whose sum is direct modulo $N$. Then $b(A_j) \neq 0$ for each $j$. If $t > m$ then for some $j$

$$\left( \sum_{i \neq j} b(A_i) \right) \cap b(A_j) \neq 0.$$  

But then

$$b(\sum_{i \neq j} A_i) \cap A_j \neq 0$$

and so $\sum_{i \neq j} A_i \nexists N$, a contradiction. Thus $\text{rank}(M/N)_R \leq m$. Hence by 1.1, $\text{rank}(M/Na_i^{-1})_R \leq m$. However, by 1.3, $\bigcap Na_i^{-1} = b(N) = 0$, and so $\text{rank } M_R \leq mn$.

As an immediate consequence, we deduce a result of Lanski [3].

COROLLARY 1.5 If $S$ is a right Goldie ring, so too is $R$.

Finally in this section we discuss the application of these results to bimodules. Suppose that $R,S$ remain as specified. There are natural ring homomorphisms...