2.0 Introduction

Direct descriptions of evolution processes by means of second order recurrences arise more frequently in physics than those by first order recurrences. An elementary example is the study of charged particles constrained to move inside toroidal surfaces, like those existing in contemporary accelerators and storage rings. When the transverse and longitudinal motions are not strongly coupled, then the non-dimensional equations of motion can be brought to the standard form

\[ \begin{align*}
X_{n+1} &= y_n + F(x_n), \\
Y_{n+1} &= -x_n + F(x_{n+1}), \quad F(0) = 0, \quad F(1) = 1,
\end{align*} \]

where \( F \) is a known smooth function. For the longitudinal motion in an alternating gradient accelerator \([G12]\)

\[ F(x) = x - \frac{1 - \mu}{\cos f_s} \sin(bx + \psi_s) - \sin \psi_s, \quad b = \pi - \psi_s, \quad 0 < \psi_s < \frac{\pi}{2}, \]

where \( \psi_s \) is the so-called equilibrium phase and \( \mu \) a parameter depending on the type of accelerated particle, the amplitude of the sinusoidal accelerating voltage and on the focusing properties of the ring. In a microtron, only one of the parameters is free, because of the operating constraint \( \psi_s = \pi/2 - \arctan \frac{1-\mu}{\pi} \). For transverse motion in one plane in the presence of thin sextupoles or octupoles \([G11]\) the function \( F(x) \) in (2-2) is replaced by

\[ F(x) = \mu x + (1 - \mu)x^2 \quad \text{and} \quad F(x) = \mu x + (1 - \mu)x^3, \]

respectively. The recurrence (2-1) has the special property that the Jacobian determinant of its right-hand sides is identically equal to unity. For this reason it is called conservative, or sometimes Hamiltonian, to reflect the fact that in an equivalent description by means of differential equations, the latter would derive from a Hamiltonian function.

A second example from physics is the analysis of open resonators by means of laws of geometrical optics. In the case of a two-dimensional resonator formed by a plane and a curved quartic mirror one encounters the non-conservative recurrence \([M3]\)

\[ \begin{align*}
U_{n+1} &= u_n - 2v_n + P_3(u_n, v_n), \\
V_{n+1} &= 2s u_n + (1 - 4s)v_n + Q_3(u_n, v_n),
\end{align*} \]

where \( a_m, b_m, s \) are constants, and \( u_n, v_n \) represent the tangent of the reflection
angle and the normalized distance between the mirrors, respectively.

In population dynamics an extension of the Nicholson-Bailey host-parasitoid relation gives rise to the non-conservative recurrence (see [B2-3] and references therein)

\[ X_{n+1} = x_n \cdot f_1(x_n)^{f_2(x_n, y_n)} \quad \text{and} \quad Y_{n+1} = y_n \cdot f_3(x_n, y_n) \]

where \( x_n \) and \( y_n \) are host and parasitoid densities. The \( f_1(x_n) \) describes the per capita rate of increase of the host as a function of its own density, \( f_2(x_n, y_n) \) the proportional survival of \( x_n \) hosts confronted by \( y_n \) parasitoids, and \( f_3(x_n, y_n) \) the per capita rate of increase of the parasitoid as a function of its own, and its hosts, densities. A basic form of (2-5), when both host and parasitoid have discrete synchronised generations is [B2]

\[ X_{n+1} = x_n \exp \left[ c \left( 1 - \frac{x_n}{K} \right) \exp(-a y_n) \right] \]

where \( a, b, c, \) and \( K \) are positive constants. \( c \) represents the intrinsic growth rate of the host and \( K \) the carrying capacity of the environment, \( a \) and \( b \) being coupling constants.

A recurrence from economics describes the Samuelson accelerator-multiplier model [S2] for the national income \( x_n \):

\[ x_n = k \cdot x_{n-1} + c(x_{n-1} - x_{n-2}) + A_n + y_n \]

where the first term represents the consumption, the second acceleration investment, the third autonomous investment and the fourth net government outlay. When the government tries to follow a contracyclical policy then \( y_n \) is made to depend on \( x_n \) or its antecedents. The simplest four cases are

\[ y_n = a \left( x_{n-1} - x_{n-2} \right) = \text{compensation of income trends} \]

\[ y_n = b \left( E - x_n \right) = \text{adjustment to a specified value } E \text{ via present income} \]

\[ y_n = b \left( E - x_{n-1} \right) = \text{adjustment to a specified value } E \text{ via past income} \]

\[ y_n = b \left( E - x_{n-2} \right) = \text{adjustment to a specified value } E \text{ via past antecedents} \]

\( E \) represents usually a desired income level (corresponding, for instance, to full employment), \( a \) is a real constant and \( b \) a positive one. An implicit assumption behind (2-7) and (2-8) is that the state of the economy is very close to a state of stable equilibrium, so that non-linear dependences do not need to be taken into account. This assumption is very convenient, but hardly realistic, at least in the contemporary economy.

The use of indirect descriptions of evolution processes by means of second order autonomous recurrences is motivated by the following two properties:

a) compared to smooth continuous formulations (for example, in terms of differential