1. Introduction

Let $D$ be a bounded domain in $\mathbb{C}^n$ (or, more generally, in a complex Banach space $E$). Let $f: D \to D$ be a holomorphic mapping. The set

$$\text{Fix } f = \{ x \in D \mid f(x) = x \}$$

has been studied by many people. Let us recall first the following theorem proved by E. Vesentini [8 and 9]:

**Theorem 1.1:** Let $B$ be the open unit ball of a complex Banach space $E$. Suppose that every point $z$ belonging to the boundary $\partial B$ of $B$ is a complex extreme point of $B$. Let $f: B \to B$ be a holomorphic mapping such that $f(0) = 0$. Then

$$\text{Fix } f = B \cap F,$$

where

$$F = \{ v \in E \mid f'(0) \cdot v = v \}$$

is the eigenspace of the derivative $f'(0)$ of $f$ at the origin for the eigenvalue 1. Moreover, if $E$ is reflexive, there exists a projection $p: E \to F$ of norm 1. So, $B \cap F$ is the image of a linear retraction $B \to B \cap F$.

The proof is based on the notion of complex geodesic. In fact, E. Vesentini [9] proved that, given $z \in D$, there exists a unique complex geodesic through the origin and $z$, and, more or less, this argument concludes the proof. But, in general, complex geodesics are not unique, and Vesentini's proof cannot be generalized. For example, the case of the bidisc $\Delta \times \Delta$ has been studied by M. Hervé [6] and E. Vesentini [8], and they proved the following result:

**Theorem 1.2:** Let $f: \Delta \times \Delta \to \Delta \times \Delta$ be a holomorphic mapping. The set $\text{Fix } f$ is one of the following sets:

1. the empty set $\emptyset$;
2. one point;
3. there exists a holomorphic mapping $\varphi: \Delta \to \Delta$ such that

$$\text{Fix } f = \{(\zeta_1, \zeta_2) \in \Delta \times \Delta \mid \zeta_2 = \varphi(\zeta_1)\}$$
or

\[ \text{Fix } f = \{(\zeta_1, \zeta_2) \in \Delta \times \Delta \mid \zeta_1 = \varphi(\zeta_2)\}; \]

4. \( \Delta \times \Delta \).

So, in this example, the set \( \text{Fix } f \) is not a linear subspace, but it is always a connected submanifold.

Now, we are going to give the results of this talk, and, first, we will begin with the finite-dimensional case.

2. Bounded domains in \( \mathbb{C}^n \)

We begin with the following result:

**Theorem 2.1:** ([13]) Let \( D \) be a bounded domain in \( \mathbb{C}^n \) and let \( f: D \to D \) be a holomorphic mapping. Then \( \text{Fix } f \) is a complex submanifold of \( D \). If \( a \in \text{Fix } f \), its tangent space \( T_a(\text{Fix } f) \) is equal to

\[ F = \{ v \in \mathbb{C}^n \mid f'(a) \cdot v = v \}. \]

The proof of this result uses ideas of H. Cartan [3] and E. Bedford [1]. Let \( a \in \text{Fix } f \), and let us consider the sequence \( f^{p} = f \circ \cdots \circ f \) (\( p \) times) of iterates of \( f \). We can find a sequence of integers \( p_j \to +\infty \) such that \( q_j = p_{j+1} - p_j \) and \( r_j = p_{j+1} - 2p_j \) converge to \( +\infty \) and that \( f^{p_j} \) converges to a holomorphic map \( F \) (uniformly on compact subsets of \( D \)). Now, by taking subsequences of the sequences \( q_j \) and \( r_j \), we can suppose that

\[ f^{q_j} \to \rho, \quad f^{r_j} \to G. \]

By shrinking \( D \) if necessary, we can suppose that \( \rho, F \) and \( G \) send \( D \) to \( D \). Then, by composition, one proves easily the following relations:

\[ \rho \circ F = F \circ \rho = F, \quad F \circ G = G \circ F = \rho, \quad f \circ \rho = \rho \circ f. \]

We deduce that

\[ \rho^2 = \rho \circ \rho = \rho \circ F \circ G = F \circ G = \rho. \]

So, \( \rho \) is a holomorphic retraction, and, by a result of H. Cartan [4], there exists a local coordinate chart \( u \) defined on a neighbourhood \( U \) of \( a \), such that \( u(a) = 0 \) and that \( u \circ \rho \circ u^{-1} \) is a linear projection.

We have proved that \( \rho(D) \) is a submanifold of \( D \) containing \( \text{Fix } f \), and it is easy to prove that \( f \) is a biholomorphic automorphism of \( \rho(D) \). It is clear that \( \rho(D) \) is a hyperbolic manifold ([5]), and we can apply the following result of H. Cartan [2]: