THE DILOGARITHM AND EXTENSIONS OF LIE ALGEBRAS

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One of the nice things about algebraic K-theory is that it is forever leading the researcher in new and unexpected directions. Several years ago I encountered in this connection the dilogarithm function

$$\int \log(1-t)\,\frac{dt}{t}\ ,$$

which was the key ingredient in a regulator map

$$K_2(X) \to H^1(X,\mathbf{C}^*)$$

for a Riemann surface X. I had various constructions for this map; a direct function theoretic approach when X was an elliptic curve in [1], a sheaf-theoretic approach in [3], and an interpretation via periods of integrals and generalized intermediate jacobians in [4]. All were complicated and in various ways unsatisfactory.

More recently Deligne found a simpler and more powerful construction based on interpreting $H^1(X,\mathbf{C}^*)$ as the group of line bundles on X with connections. This fitted nicely with an idea of Rama krishnan that the dilogarithm could be interpreted as a single valued map

(0.1) $$\mathbf{P}^1_{\mathbf{C}} - \{0,1,\infty\} \to H(\mathbf{Z})\backslash H(\mathbf{C})$$

where H(R) for any ring R is the Heisenberg group of unipotent 3×3 upper triangular matrices. $H(\mathbf{Z})\backslash H(\mathbf{C})$ has a natural structure of principal bundle over $\mathbf{C}^* \times \mathbf{C}^*$ with fiber $\mathbf{C}^*$. Moreover this bundle has a standard (non-integrable) holomorphic connection ∇. A symbol $\{f,g\} \in K_2(X)$ corresponds to a map

$$(f,g): X \to \mathbf{C}^* \times \mathbf{C}^*$$

(ignore for the moment the problem of zeroes and poles of f and g as well as the question of whether f and g are well-defined) so we may associate to $\{f,g\}$ the bundle

$$(f,g)^*(H(\mathbf{Z})\backslash H(\mathbf{C}),\nabla) \in H^1(X,\mathbf{C}^*).$$

That this works and leads to a regulator map is the content of §1, Rama krishnan's map (0.1) above gives the Steinberg relation

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$$(f,1-f)^*(H(\mathbf{Z})\backslash H(\mathbf{C}),\nabla) = (e).$$

As an application we answer in (1.24) a question of Tate ([16], p.250] concerning torsion in K_2.

In communicating to me his construction, Deligne remarked (cryptically, as is his wont) that he had found it while thinking about Kac-Moody Lie algebras. Aspects of that relationship (Another new and unexpected direction!) are discussed in §2. Taking $X = \mathrm{Spec}(R)$ to be an affine Riemann surface, the regulator map leads to an extension

$$0 \to H^1(X,\mathbf{C}^*) \to W \to SL(R) \to 1 \tag{0.2}$$

It then turns out that one can associate to (0.2) a central extension of lie algebras

$$0 \to H^1(X,\mathbf{C}) \to \mathfrak{g} \to sl(R) \to 0 \tag{0.3}$$

given by the lie algebra cocycle

$$A,B \longmapsto \mathrm{Trace}(A\cdot dB) \in H^1(X,\mathbf{C}). \tag{0.4}$$

In fact (0.3) is the universal central extension of $sl(R)$ in the category of lie algebras over $\mathbf{C}$. More generally, if k is a commutative ring with $\frac{1}{2} \in k$ and R is a k-algebra (commutative with 1) then

$$H_2(sl(R),k) \cong \Omega^1_{R/k}/dR = \text{Kähler 1-forms mod exact 1-forms},$$

and the cocycle (0.4) leads to the universal central extension (over k)

$$0 \to \Omega^1/dR \to \mathfrak{g} \to sl(R) \to 0.$$

The proof of this is a straightforward application of ideas of Steinberg and is given in §3. It would be exciting to know the groups $H_*(sl(R),k)$ for $* > 2$ and to understand their relation to higher regulators. (A technical point: it might be better to work with the relative groups $H_*(sl(R),sl(k);k)$. This is closer in spirit to the calculations which have been done for Kac-Moody algebras [9],[10],[12], and avoids a good deal of garbage which would otherwise appear in H_3.) In fact, based on the ideas in [9] and [12], it seems natural to conjecture when $k = \mathbf{C}$

$$\varinjlim_n H_*(sl_n(R),sl_n(\mathbf{C});\mathbf{C}) \cong \varinjlim_n H_*((U_n(\mathbf{C}),e)^{(X,x_0)},\mathbf{C})_{top.sp.}$$

$$\cong \bigotimes_g \varinjlim_n H_*(\Omega U_n(\mathbf{C})/U_n(\mathbf{C}),\mathbf{C})_{top.sp.}$$